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Anisotropic Landau-Lifshitz sigma models from q -deformed $\text{AdS}_5 \times \text{S}^5$ superstrings

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ABSTRACT: We consider bosonic subsectors of the q -deformed $\text{AdS}_5 \times \text{S}^5$ superstring action and study the classical integrable structure of anisotropic Landau-Lifshitz sigma models (LLSMs) derived by taking fast-moving string limits. The subsectors are 1) deformed $\text{AdS}_3 \times \text{S}^1$ and 2) $\text{R} \times$ deformed S^3 . The cases 1) and 2) lead to a time-like warped $SL(2)$ LLSM and a squashed S^3 LLSM, respectively. For each of them, we construct an infinite number of non-local conserved charges and show a quantum affine algebra at the classical level. Furthermore, a pp-wave like limit is applied for the case 1). The resulting system is a null-like warped $SL(2)$ LLSM and exhibits a couple of Yangians through non-local gauge transformations associated with Jordanian twists.

KEYWORDS: AdS-CFT Correspondence, Sigma Models, Integrable Field Theories

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1 Introduction

The AdS/CFT correspondence [1] is a particular class of dualities between string (gravity) theories and gauge theories. It is firmly supported by an enormous amount of works to date and its various aspects have been elucidated. In particular, the discovery of an integrable structure behind it [2] is a triumph of modern theoretical and mathematical physics.

A significant feature of AdS/CFT is that the $\text{AdS}_5 \times \text{S}^5$ background is represented by a supercoset

$$\text{AdS}_5 \times \text{S}^5 = \frac{PSU(2, 2|4)}{SO(1, 4) \times SO(5)}. \quad (1.1)$$

It enables us to construct the Green-Schwarz string action [3] in terms of group elements and the string sigma model is classically integrable [4]. The supercoset (1.1) enjoys the \mathbb{Z}_4 -grading that ensures the existence of an infinite number of conserved charges [5–8]. Other possible cosets are classified from the consistency conditions for string backgrounds [9].

The next is to consider integrable deformations of AdS/CFT. There are two approaches to tackle this issue. The one is an algebraic approach based on q -deformations of the world-sheet S-matrix [10–14]. The deformed S-matrices are constructed in a mathematically consistent way. The other is a geometric approach to argue deformations of target spaces of string sigma models. Integrable deformations of two-dimensional non-linear sigma models have a long history (For classic papers, see [15–17]). In the present, there is a renewed interest in this subject in relation to the study of AdS/CFT.

We will focus upon the latter approach here. Deformed target spaces are not represented by symmetric cosets, but typically by non-symmetric cosets (For arguments on some examples, see [18]). A particularly simple and tractable example is squashed S^3 and its integrable structure has been studied intensively [19–28]. In the recent, a generalization to higher dimensions has succeeded for arbitrary compact Lie groups and symmetric cosets [29], by following the Yang-Baxter sigma model description [30].

Just after that, a standard q -deformed $\text{AdS}_5 \times \text{S}^5$ superstring action has been constructed with a linear R-operator of Drinfeld-Jimbo type [31–33] satisfying the modified classical Yang-Baxter equation [34]. Then the metric (in the string frame) and NS-NS two-form have been determined [35]. Some special cases of the background are examined in [36] and a mirror TBA is proposed in [37].

On the other hand, there is another kind of q -deformations called Jordanian deformations [38–40]. Jordanian deformed $\text{AdS}_5 \times \text{S}^5$ superstring actions have been constructed with linear R-operators satisfying classical Yang-Baxter equation [41]. A remarkable point

is that partial deformations are possible in comparison to the standard q -deformation. In fact, as an example of deformation of AdS_5 , a complete type IIB supergravity solution has been found in [42] and it contains a three-dimensional Schrödinger spacetime (Sch_3) as a subspace. Furthermore, γ -deformed backgrounds [43, 44] and the gravity duals for non-commutative gauge theories [45–47] have been reproduced in [48] and [49], respectively, in the context of the Yang-Baxter sigma model.

In this paper, we will concentrate on the standard q -deformation of the $\text{AdS}_5 \times \text{S}^5$ superstring constructed in [34]. A serious problem is that the metric introduced in [35] is singular. In order to resolve the singularity, it would be a nice way to consider a non-relativistic limit of the string world-sheet. It is performed by considering fast-moving strings [50–53]. For simplicity, we consider two subsectors, 1) deformed $\text{AdS}_3 \times \text{S}^1$ and 2) $\text{R} \times$ deformed S^3 . The deformed AdS_3 is still singular and hence it is enough to take this subspace so as to argue how to avoid the singularity.

There is another advantage of taking a non-relativistic limit from the viewpoint of the classical integrable structure. While non-ultra local terms appear in the current algebra in the relativistic case, those do not appear in the non-relativistic case. Hence infinite-dimensional symmetries generated by conserved non-local charges can be studied in a definite manner without ambiguities.

This paper is organized as follows. Section 2 considers fast-moving string limits of the deformed $\text{AdS}_3 \times \text{S}^1$ and $\text{R} \times$ deformed S^3 subsectors of the standard q -deformed $\text{AdS}_5 \times \text{S}^5$. The resulting systems are a time-like warped $SL(2)$ LLSM and a squashed S^3 LLSM, respectively. Section 3 reveals the classical integrable structure of the time-like warped $SL(2)$ LLSM. The Lax pair is constructed and a (classical analogue of) quantum affine algebra $U_q(\widehat{sl(2)})$ is presented. The classical r -matrix is of trigonometric type. In section 4 we study a pp-wave like limit of the time-like warped $SL(2)$ LLSM. Then the resulting system is a null-like warped $SL(2)$ LLSM. A direct computation leads to an exotic symmetry as in [24] and the classical r -matrix contains a deformation term. However, non-local gauge transformations can be performed by following [25] so as to undo Jordanian twists. As a result, a couple of Yangians $\mathcal{Y}(sl(2))$ are derived and the resulting r -matrix is of rational. Section 5 is devoted to conclusion and discussion.

In Appendix A, our convention of the $sl(2)$ and $su(2)$ generators are summarized. Appendix B explains the classical integrable structure of the squashed S^3 LLSM. Also in this case, a quantum affine algebra $U_q(\widehat{su(2)})$ is exhibited. The classical r -matrix is of trigonometric type again. In Appendix C, we argue the relation between boundary conditions and conserved non-local charges. In Appendix D, a null-like warped LLSM is derived from a time-like LLSM via a pp-wave like limit. In Appendix E, we give the detailed computation of non-local gauge transformations in undoing Jordanian twists.

2 Fast-moving string limits of q -deformed $\text{AdS}_5 \times \text{S}^5$

In this section, we first introduce the bosonic part of the q -deformed $\text{AdS}_5 \times \text{S}^5$ superstring action constructed in [34, 35]. Then it is truncated to the following subsectors: 1) deformed

$\text{AdS}_3 \times \text{S}^1$ and 2) $\text{R} \times \text{deformed } \text{S}^3$. By taking a fast-moving string limit, each of them leads to an anisotropic LLSM.

2.1 The q -deformed $\text{AdS}_5 \times \text{S}^5$ background

The standard q -deformation of the $\text{AdS}_5 \times \text{S}^5$ superstring has been constructed in [34]. Here we are interested in the bosonic part of the deformed action, where the metric (in the string frame) and NS-NS two-form have been determined in [35].

The bosonic action is composed of the metric part S_G and the Wess-Zumino (WZ) term S_{WZ} that describes the coupling to an NS-NS two-form as follows:

$$S = S_G + S_{\text{WZ}}, \quad (2.1)$$

$$S_G = \int d\tau d\sigma [\mathcal{L}_{\text{AdS}}^G + \mathcal{L}_S^G], \quad S_{\text{WZ}} = \int d\tau d\sigma [\mathcal{L}_{\text{AdS}}^{\text{WZ}} + \mathcal{L}_S^{\text{WZ}}].$$

Here S_G and S_{WZ} are divided into the AdS part and the internal sphere part.

The metric parts $\mathcal{L}_{\text{AdS}}^G$ and \mathcal{L}_S^G are given by, respectively,

$$\mathcal{L}_{\text{AdS}}^G = -\frac{T_{(\varkappa)}}{2} \eta^{\mu\nu} \left[-\frac{(1+\rho^2) \partial_\mu t \partial_\nu t}{1-\varkappa^2 \rho^2} + \frac{\partial_\mu \rho \partial_\nu \rho}{(1+\rho^2)(1-\varkappa^2 \rho^2)} + \frac{\rho^2 \partial_\mu \zeta \partial_\nu \zeta}{1+\varkappa^2 \rho^4 \sin^2 \zeta} \right. \\ \left. + \frac{\rho^2 \cos^2 \zeta \partial_\mu \psi_1 \partial_\nu \psi_1}{1+\varkappa^2 \rho^4 \sin^2 \zeta} + \rho^2 \sin^2 \zeta \partial_\mu \psi_2 \partial_\nu \psi_2 \right], \quad (2.2)$$

$$\mathcal{L}_S^G = -\frac{T_{(\varkappa)}}{2} \eta^{\mu\nu} \left[\frac{(1-r^2) \partial_\mu \phi \partial_\nu \phi}{1+\varkappa^2 r^2} + \frac{\partial_\mu r \partial_\nu r}{(1-r^2)(1+\varkappa^2 r^2)} + \frac{r^2 \partial_\mu \xi \partial_\nu \xi}{1+\varkappa^2 r^4 \sin^2 \xi} \right. \\ \left. + \frac{r^2 \cos^2 \xi \partial_\mu \phi_1 \partial_\nu \phi_1}{1+\varkappa^2 r^4 \sin^2 \xi} + r^2 \sin^2 \xi \partial_\mu \phi_2 \partial_\nu \phi_2 \right]. \quad (2.3)$$

Here the deformed AdS_5 part is parameterized by the coordinates $t, \psi_1, \psi_2, \zeta, \rho$. The deformed S^5 part is described by $\phi, \phi_1, \phi_2, \xi, r$. The world-sheet metric $\eta_{\mu\nu}$ is taken as the flat metric $\eta_{\mu\nu} = (-1, +1)$ with the world-sheet coordinates $\sigma^\mu = (\sigma^0, \sigma^1) = (\tau, \sigma)$. The periodic boundary condition is imposed for the σ -direction and the range is taken as $\sigma \in [0, 2\pi]$.

Then the deformation is measured by a real parameter $\varkappa \in [0, \infty)$. This parameter can also be expressed in terms of another real parameter $\eta \in [0, 1)$ like

$$\varkappa = \frac{2\eta}{1-\eta^2}. \quad (2.4)$$

When $\varkappa = 0$, the above action is reduced to the undeformed $\text{AdS}_5 \times \text{S}^5$ one. The range of the coordinates is the same as that of the ones with $\varkappa = 0$. A \varkappa -dependent string tension $T_{(\varkappa)}$ is defined as

$$T_{(\varkappa)} \equiv T(1+\varkappa^2)^{1/2}, \quad T \equiv \frac{R^2}{2\pi\alpha'}. \quad (2.5)$$

Here T is a dimensionless string tension with R^2 , where R is the AdS radius when $\varkappa = 0$.

Finally the WZ parts are given by, respectively,

$$\mathcal{L}_{\text{AdS}}^{\text{WZ}} = \frac{T(\varkappa)}{2} \varkappa \epsilon^{\mu\nu} \frac{\rho^4 \sin 2\zeta}{1 + \varkappa^2 \rho^4 \sin^2 \zeta} \partial_\mu \psi_1 \partial_\nu \zeta, \quad (2.6)$$

$$\mathcal{L}_S^{\text{WZ}} = -\frac{T(\varkappa)}{2} \varkappa \epsilon^{\mu\nu} \frac{r^4 \sin 2\xi}{1 + \varkappa^2 r^4 \sin^2 \xi} \partial_\mu \phi_1 \partial_\nu \xi, \quad (2.7)$$

where $\epsilon^{\mu\nu}$ is the totally anti-symmetric tensor on the string world-sheet and it is normalized as $\epsilon^{01} = +1$. The WZ parts are proportional to κ , and hence vanish when $\varkappa = 0$.

A deformed $\text{AdS}_3 \times \text{S}^3$ subspace

For later purpose, we consider a deformed $\text{AdS}_3 \times \text{S}^3$ subspace of the q -deformed $\text{AdS}_5 \times \text{S}^5$. By imposing the following conditions,

$$\zeta = 0, \quad \xi = 0, \quad (2.8)$$

S^G is restricted to a deformed $\text{AdS}_3 \times \text{S}^3$. The metric parts are given by

$$\mathcal{L}_{\text{AdS}_3}^G = -\frac{T(\varkappa)}{2} \eta^{\mu\nu} \left[-\frac{(1 + \rho^2) \partial_\mu t \partial_\nu t}{1 - \varkappa^2 \rho^2} + \frac{\partial_\mu \rho \partial_\nu \rho}{(1 + \rho^2)(1 - \varkappa^2 \rho^2)} + \rho^2 \partial_\mu \psi_1 \partial_\nu \psi_1 \right], \quad (2.9)$$

$$\mathcal{L}_{\text{S}^3}^G = -\frac{T(\varkappa)}{2} \eta^{\mu\nu} \left[\frac{(1 - r^2) \partial_\mu \phi \partial_\nu \phi}{1 + \varkappa^2 r^2} + \frac{\partial_\mu r \partial_\nu r}{(1 - r^2)(1 + \varkappa^2 r^2)} + r^2 \partial_\mu \phi_1 \partial_\nu \phi_1 \right]. \quad (2.10)$$

Note that the S_{WZ} part vanishes under the condition (2.8).

To take fast-moving string limits, it is helpful to perform coordinate transformations,

$$\rho \rightarrow \sinh \rho, \quad r \rightarrow \sin \theta. \quad (2.11)$$

Then the metric parts (2.9) and (2.10) are rewritten as

$$\mathcal{L}_{\text{AdS}_3}^G = -\frac{T(\varkappa)}{2} \eta^{\mu\nu} \left[-\frac{\cosh^2 \rho \partial_\mu t \partial_\nu t}{1 - \varkappa^2 \sinh^2 \rho} + \frac{\partial_\mu \rho \partial_\nu \rho}{1 - \varkappa^2 \sinh^2 \rho} + \sinh^2 \rho \partial_\mu \psi_1 \partial_\nu \psi_1 \right], \quad (2.12)$$

$$\mathcal{L}_{\text{S}^3}^G = -\frac{T(\varkappa)}{2} \eta^{\mu\nu} \left[\frac{\cos^2 \theta \partial_\mu \phi \partial_\nu \phi}{1 + \varkappa^2 \sin^2 \theta} + \frac{\partial_\mu \theta \partial_\nu \theta}{1 + \varkappa^2 \sin^2 \theta} + \sin^2 \theta \partial_\mu \phi_1 \partial_\nu \phi_1 \right]. \quad (2.13)$$

This truncated action will be the starting point of our later arguments.

2.2 A fast-moving string limit of deformed $\text{AdS}_3 \times \text{S}^1$

Let us consider the string action on a deformed $\text{AdS}_3 \times \text{S}^1$ subsector of the truncated action. By setting that $\theta = 0$, the Lagrangian is given by

$$\begin{aligned} \mathcal{L}_{\text{AdS}_3 \times \text{S}^1}^G = & -\frac{T(\varkappa)}{2} \eta^{\mu\nu} \left[-\frac{\cosh^2 \rho \partial_\mu t \partial_\nu t}{1 - \varkappa^2 \sinh^2 \rho} + \frac{\partial_\mu \rho \partial_\nu \rho}{1 - \varkappa^2 \sinh^2 \rho} \right. \\ & \left. + \sinh^2 \rho \partial_\mu \psi_1 \partial_\nu \psi_1 + \partial_\mu \phi \partial_\nu \phi \right], \end{aligned} \quad (2.14)$$

where an S^1 circle is described by ϕ .

To derive an LLSM from this subsector, let us perform a coordinate transformation,

$$\psi_1 = \tilde{\psi} + t, \quad \phi = \tilde{\phi} + t, \quad \rho = \frac{\tilde{\rho}}{2}. \quad (2.15)$$

Then the Lagrangian (2.14) is rewritten as

$$\begin{aligned} \mathcal{L}_{\text{AdS}_3 \times \text{S}^1}^G = & -\frac{T_{(\varkappa)}}{2} \eta^{\mu\nu} \left[-\frac{\varkappa^2 \sinh^2 \frac{\tilde{\rho}}{2} \cosh^2 \frac{\tilde{\rho}}{2}}{1 - \varkappa^2 \sinh^2 \frac{\tilde{\rho}}{2}} \partial_\mu t \partial_\nu t + 2 \left(\sinh^2 \frac{\tilde{\rho}}{2} \partial_\mu \tilde{\psi} + \partial_\mu \tilde{\phi} \right) \partial_\nu t \right. \\ & \left. + \frac{1}{4} \frac{\partial_\mu \tilde{\rho} \partial_\nu \tilde{\rho}}{1 - \varkappa^2 \sinh^2 \frac{\tilde{\rho}}{2}} + \sinh^2 \frac{\tilde{\rho}}{2} \partial_\mu \tilde{\psi} \partial_\nu \tilde{\psi} + \partial_\mu \tilde{\phi} \partial_\nu \tilde{\phi} \right]. \end{aligned} \quad (2.16)$$

With the static gauge

$$t = \kappa \tau,$$

the following fast-moving string limit is taken,

$$\varkappa \rightarrow 0, \quad \dot{X}^\mu \rightarrow 0, \quad \kappa \rightarrow \infty \quad \text{with} \quad \kappa \varkappa, \quad \kappa \dot{X}^\mu : \text{ fixed}. \quad (2.17)$$

Note that the above limit (2.17) contains a further condition on $\kappa \varkappa$ as well as the standard one discussed in [50, 52, 53].

After all, the resulting action is given by

$$\begin{aligned} S = \frac{T}{2} \int d\tau d\sigma \left[-\frac{1}{4} \kappa^2 \varkappa^2 \sinh^2 \tilde{\rho} + \kappa [(\cosh \tilde{\rho} - 1) \dot{\tilde{\psi}} + 2\dot{\tilde{\phi}}] \right. \\ \left. - \frac{\dot{\tilde{\rho}}^2}{4} - \sinh^2 \frac{\tilde{\rho}}{2} \dot{\tilde{\psi}}^2 - \dot{\tilde{\phi}}^2 \right]. \end{aligned} \quad (2.18)$$

Here $T_{(\varkappa)}$ has been reduced to the dimensionless tension T after performing the limit (2.17). A remarkable point is that the system (2.18) has no singular term in comparison to the original metric.

The Virasoro constraints are also rewritten under the limit (2.17). To the leading order in κ , the one of the Virasoro constraints becomes

$$0 = \kappa [(\cosh \tilde{\rho} - 1) \dot{\tilde{\psi}} + 2\dot{\tilde{\phi}}]. \quad (2.19)$$

By eliminating $\dot{\tilde{\phi}}$ from (2.18) with (2.19), the leading-order action is given by

$$\begin{aligned} S = \frac{T}{2} \int d\tau d\sigma \left[-\frac{1}{4} \kappa^2 \varkappa^2 \sinh^2 \tilde{\rho} + \kappa [(\cosh \tilde{\rho} - 1) \dot{\tilde{\psi}} + 2\dot{\tilde{\phi}}] \right. \\ \left. - \frac{1}{4} (\dot{\tilde{\rho}}^2 + \sinh^2 \tilde{\rho} \dot{\tilde{\psi}}^2) \right]. \end{aligned} \quad (2.20)$$

The above action is simple and contains just a single deformation term (only the first term).

A comparison to the time-like warped AdS₃ case

It is valuable to rewrite the action (2.20) by introducing new parameters,

$$C \equiv \frac{\varkappa^2}{4} \geq 0, \quad L \equiv \frac{R^2 \kappa}{2\pi\alpha'}, \quad \lambda \equiv \frac{R^4}{\alpha'^2}. \quad (2.21)$$

Then the action (2.20) is rewritten as

$$S = \frac{L}{2} \int dt d\sigma \left[-C \cosh^2 \tilde{\rho} + [(\cosh \tilde{\rho} - 1) \partial_t \tilde{\psi} + 2 \partial_t \tilde{\phi}] \right. \\ \left. - \frac{\lambda}{16\pi^2 L^2} [(\partial_\sigma \tilde{\rho})^2 + \sinh^2 \tilde{\rho} (\partial_\sigma \tilde{\psi})^2] \right]. \quad (2.22)$$

Here τ is replaced by t through $t = \kappa\tau$. The deformation term has also been rewritten like $-C \sinh^2 \tilde{\rho} \rightarrow -C \cosh^2 \tilde{\rho} + C$ and then the constant term C has been dropped off.

As a result, the derived action (2.22) agrees precisely with a fast-moving string limit of time-like warped AdS₃ × S¹ string sigma model [54]. Hence we call it *time-like warped SL(2) LLSM*.

2.3 A fast-moving string limit of R × deformed S³

We next consider the string action on an R × deformed S³ subsector of the truncated action. By setting that $\rho = 0$, the Lagrangian is given by

$$\mathcal{L}_{R \times S^3}^G = -\frac{T_{(\varkappa)}}{2} \eta^{\mu\nu} \left[-\partial_\mu t \partial_\nu t + \frac{\cos^2 \theta \partial_\mu \phi \partial_\nu \phi}{1 + \varkappa^2 \sin^2 \theta} + \frac{\partial_\mu \theta \partial_\nu \theta}{1 + \varkappa^2 \sin^2 \theta} + \sin^2 \theta \partial_\mu \phi_1 \partial_\nu \phi_1 \right]. \quad (2.23)$$

Here the time coordinate in the AdS part is included as R. It should be remarked that the reduced system (2.23) is not singular due to the condition $\rho = 0$, even though the time direction has been included.

As in the previous subsection, let us first perform the coordinate transformation,

$$\phi = \varphi_1 + \varphi_2, \quad \phi_1 = \varphi_1 - \varphi_2, \quad \varphi_1 = t + \tilde{\varphi}_1. \quad (2.24)$$

Then the Lagrangian (2.23) is rewritten as

$$\mathcal{L}_{R \times S^3}^G = -\frac{T_{(\varkappa)}}{2} \eta^{\mu\nu} \left[-\frac{\varkappa^2 \sin^2 \theta \cos^2 \theta}{1 + \varkappa^2 \sin^2 \theta} \partial_\mu t \partial_\nu t + \left(1 - \frac{\varkappa^2 \sin^2 \theta}{1 + \varkappa^2 \sin^2 \theta} \right) \partial_\mu \theta \partial_\nu \theta \right. \\ \left. + 2 \left\{ \left(1 - \frac{\varkappa^2 \sin^2 \theta \cos^2 \theta}{1 + \varkappa^2 \sin^2 \theta} \right) \partial_\mu \tilde{\varphi}_1 + \left(\cos 2\theta - \frac{\varkappa^2 \sin^2 \theta \cos^2 \theta}{1 + \varkappa^2 \sin^2 \theta} \right) \partial_\mu \varphi_2 \right\} \partial_\nu t \right. \\ \left. + \left(1 - \frac{\varkappa^2 \sin^2 \theta \cos^2 \theta}{1 + \varkappa^2 \sin^2 \theta} \right) (\partial_\mu \tilde{\varphi}_1 \partial_\nu \tilde{\varphi}_1 + \partial_\mu \varphi_2 \partial_\nu \varphi_2) \right. \\ \left. + 2 \left(\cos 2\theta - \frac{\varkappa^2 \sin^2 \theta \cos^2 \theta}{1 + \varkappa^2 \sin^2 \theta} \right) \partial_\mu \tilde{\varphi}_1 \partial_\nu \varphi_2 \right]. \quad (2.25)$$

This Lagrangian is drastically simplified with the static gauge

$$t = \kappa\tau$$

and by taking the fast-moving string limit (2.17). The resulting action is

$$S = \frac{T}{2} \int d\tau d\sigma \left[-\frac{1}{4} \kappa^2 \varkappa^2 \sin^2 2\theta + 2\kappa [\dot{\tilde{\varphi}}_1 + \cos 2\theta \dot{\varphi}_2] - \dot{\theta}^2 - \dot{\varphi}_1^2 - \dot{\varphi}_2^2 - 2 \cos 2\theta \dot{\varphi}_1 \dot{\varphi}_2 \right]. \quad (2.26)$$

Here $T_{(\varkappa)}$ is replaced by T after taking the limit (2.17), again.

The Virasoro constraints are also changed under the limit (2.17). To the leading order in κ , the one of the Virasoro constraints is rewritten as

$$0 = 2\kappa [\dot{\tilde{\varphi}}_1 + \cos 2\theta \dot{\varphi}_2]. \quad (2.27)$$

By eliminating $\dot{\tilde{\varphi}}_1$ from (2.26) with (2.27), the leading-order action is given by

$$S = \frac{T}{2} \int d\tau d\sigma \left[-\frac{1}{4} \kappa^2 \varkappa^2 \sin^2 2\theta + 2\kappa [\dot{\tilde{\varphi}}_1 + \cos 2\theta \dot{\varphi}_2] - \dot{\theta}^2 - \sin^2 2\theta \dot{\varphi}_2^2 \right]. \quad (2.28)$$

The resulting action is very simplified again.

A comparison to the squashed S^3 case

It is worth rewriting the action (2.28) by introducing new variables,

$$\varphi_2 = -\frac{1}{2} \tilde{\phi}, \quad \theta = \frac{1}{2} \tilde{\theta}, \quad (2.29)$$

and new parameters

$$C \equiv \frac{\varkappa^2}{4} \geq 0, \quad L \equiv \frac{R^2 \kappa}{2\pi \alpha'}, \quad \lambda \equiv \frac{R^4}{\alpha'^2}. \quad (2.30)$$

Then the action is rewritten as

$$S = \frac{L}{2} \int dt d\sigma \left[C \cos^2 \tilde{\theta} - [\cos \tilde{\theta} \partial_t \tilde{\phi} - 2\partial_t \tilde{\varphi}_1] - \frac{\lambda}{16\pi^2 L^2} [(\partial_\sigma \tilde{\theta})^2 + \sin^2 \tilde{\theta} (\partial_\sigma \tilde{\phi})^2] \right]. \quad (2.31)$$

Here τ is replaced by t through $t = \kappa\tau$. The deformation term has been rewritten as $-C \sin^2 \tilde{\theta} \rightarrow C \cos^2 \tilde{\theta} - C$ and then the constant term $-C$ has been dropped off.

Finally, the resulting action (2.31) agrees exactly with a fast-moving string limit of $R \times$ squashed S^3 string sigma model [55]. Hence we refer to it as the *squashed S^3 LLSM*.

3 Integrability of time-like warped $SL(2)$ LLSM

Let us argue the classical integrability of the time-like warped $SL(2)$ LLSM. The related infinite-dimensional symmetry is also discussed by explicitly constructing an infinite number of conserved non-local charges. In section 2 the periodic boundary condition has been imposed for the spatial direction of the string world-sheet. In the following sections, however, the string world-sheet is supposed to be spatially infinite in order to argue an infinite-dimensional symmetry based on non-local charges. In fact, the fast-moving string limit implies a decompactification limit of the σ -direction (For example, see the argument in 2.2 of [56]). For the case of the squashed S^3 LLSM, see Appendix B.

3.1 The classical action and Lax pair

The classical action of the time-like warped $SL(2)$ LLSM is given by

$$S = \frac{L}{2} \int_{-\infty}^{\infty} dt dx \left[-C \cosh^2 \rho + (\cosh \rho - 1) \partial_t \psi - \frac{\lambda}{16\pi^2 L^2} \left[(\partial_x \rho)^2 + \sinh^2 \rho (\partial_x \psi)^2 \right] \right]. \quad (3.1)$$

The world-sheet is a (1+1)-dimensional spacetime spanned by t and x and the spatial direction is infinite. The system is non-relativistic because the action contains the first order in time derivative and the second order in spatial derivative. The deformation parameter C is restricted to $C \geq 0$. When $C = 0$, an isotropic $SL(2)$ LLSM is reproduced.

It is convenient to introduce a vector representation n^a ,

$$n^0 = -\cosh \rho, \quad n^1 = \sinh \rho \sin \psi, \quad n^2 = \sinh \rho \cos \psi, \quad (3.2)$$

satisfying the following relation,

$$(n^0)^2 - (n^1)^2 - (n^2)^2 = 1. \quad (3.3)$$

Then the classical equations of motion are rewritten as

$$\begin{aligned} \partial_t n^0 &= \frac{\lambda}{8\pi^2 L^2} (n^1 \partial_x^2 n^2 - n^2 \partial_x^2 n^1), \\ \partial_t n^1 &= \frac{\lambda}{8\pi^2 L^2} (n^0 \partial_x^2 n^2 - n^2 \partial_x^2 n^0) - 2C n^0 n^2, \\ \partial_t n^2 &= \frac{\lambda}{8\pi^2 L^2} (n^1 \partial_x^2 n^0 - n^0 \partial_x^2 n^1) + 2C n^0 n^1. \end{aligned} \quad (3.4)$$

These are identical to the Landau-Lifshitz equations

$$\partial_t n_a = \varepsilon_{abc} n^b \left(\frac{\lambda}{8\pi^2 L^2} \partial_x^2 n^c + \mathcal{J}_d n^d \right) \quad (a = 0, 1, 2), \quad (3.5)$$

with an anisotropic matrix \mathcal{J}

$$\mathcal{J}_b^a = \text{diag}(j + 2C, j, j) \quad (j : \text{an arbitrary const.}). \quad (3.6)$$

Here we have introduced the totally anti-symmetric tensor ε_{abc} with $\varepsilon_{012} = -1$. The $sl(2)$ indices are raised and lowered with $\gamma^{ab} = \text{diag}(-1, +1, +1)$ and its inverse, respectively.

Lax pair. The Lax pair of the time-like warped $SL(2)$ LLSM is represented by [57]

$$\begin{aligned} U(t, x; z) &= \frac{i\alpha}{\sinh z} [-\cosh z n^0 T^0 + n^1 T^1 + n^2 T^2], \\ V(t, x; z) &= \frac{i\beta}{\sinh z} [-\cosh z (n^1 \partial_x n^2 - n^2 \partial_x n^1) T^0 + (n^0 \partial_x n^2 - n^2 \partial_x n^0) T^1 \\ &\quad + (n^1 \partial_x n^0 - n^0 \partial_x n^1) T^2] \\ &\quad + \frac{\alpha\beta}{\sinh^2 z} [-n^0 T^0 + \cosh z n^1 T^1 + \cosh z n^2 T^2], \end{aligned} \quad (3.7)$$

with a spectral parameter $z \in \mathbb{C}$ and new parameters,

$$\alpha \equiv \frac{4\pi L}{\sqrt{\lambda}} \sqrt{C}, \quad \beta \equiv \frac{\sqrt{\lambda}}{2\pi L} \sqrt{C}. \quad (3.8)$$

The equations of motion (3.4) are reproduced from the commutation relation,

$$\left[\partial_t - V(t, x; z), \partial_x - U(t, x; z) \right] = 0. \quad (3.9)$$

Monodromy matrix. The monodromy matrix is defined as

$$M(z) \equiv \text{P exp} \left[\int_{-\infty}^{\infty} dx U(t, x; z) \right], \quad (3.10)$$

where P denotes the path ordering. Due to the condition (3.9), $M(z)$ is a conserved quantity,

$$\frac{d}{dt} M(z) = 0. \quad (3.11)$$

Thus the expansion of $M(z)$ in terms of z leads to an infinite number of conserved charges. The resulting algebra of the charges depends on the expansion point. For example, the expansions around $z = \pm\infty$ leads to a quantum affine algebra $U_q(\widehat{sl(2)})$. We will elaborate a classical realization of it in the next subsection.

3.2 The standard q -deformation of $sl(2)$

We will show that a q -deformed $sl(2)$ is realized in the time-like warped $SL(2)$ LLSM.

While the $SL(2)$ symmetry is realized in an isotropic $SL(2)$ LLSM, it is broken to $U(1)$ due to the non-vanishing C . The remaining $U(1)$ charge is given by

$$Q^0 = -\frac{L}{2} \int_{-\infty}^{\infty} dx n^0(x). \quad (3.12)$$

Here it is worth noting that the broken components of $SL(2)$, 1 and 2 are still realized as non-local symmetries even when $C \neq 0$, as in the original system before taking the fast-moving string limit.

In order to show this fact, it is convenient to introduce $n^{\hat{\pm}}$ defined as

$$n^{\hat{\pm}} \equiv \frac{n^1 \pm i n^2}{\sqrt{2}}. \quad (3.13)$$

For the conservation of non-local charges, boundary conditions are sensitive. We take a rapidly damping condition so that $n^{\hat{\pm}}$ vanish at the spatial infinities. This condition enables us to construct conserved non-local charges, as shown in Appendix C.

The conserved non-local charges are given by

$$Q^{\hat{\pm}} = -\frac{L}{2} \int_{-\infty}^{\infty} dx e^{\alpha x(x)} n^{\hat{\pm}}(x), \quad (3.14)$$

where $\chi(x)$ is a non-local field defined as

$$\chi(x) \equiv \frac{1}{2} \int_{-\infty}^{\infty} dy \, \epsilon(x-y) n^0(y), \quad (3.15)$$

and $\epsilon(x)$ is the signature function defined as

$$\epsilon(x) \equiv \theta(x) - \theta(-x) \quad (3.16)$$

with the step function $\theta(x)$. It is helpful to introduce the following relations:

$$\partial_x \chi = n^0, \quad \partial_t \chi = i \frac{\lambda}{8\pi^2 L^2} \left(n^{\hat{+}} \partial_x n^{\hat{-}} - n^{\hat{-}} \partial_x n^{\hat{+}} \right). \quad (3.17)$$

Then the next is to compute the Poisson brackets of Q^0 and $Q^{\hat{\pm}}$. The Poisson brackets for n^a ($a = 0, \hat{\pm}$) are given by

$$\left\{ n^a(x), n^b(y) \right\}_{\text{P}} = -i \frac{2}{L} \epsilon^{ab}_c n^c(x) \delta(x-y). \quad (3.18)$$

Note that non-ultra local terms are not contained in comparison to the current algebra of principal chiral models.

With the brackets (3.18), the brackets of Q^0 and $Q^{\hat{\pm}}$ can be evaluated as

$$\begin{aligned} \left\{ Q^{\hat{\pm}}, Q^0 \right\}_{\text{P}} &= \pm i Q^{\hat{\pm}}, \\ \left\{ Q^{\hat{+}}, Q^{\hat{-}} \right\}_{\text{P}} &= i \frac{L}{2\alpha} \sinh\left(\frac{2\alpha}{L} Q^0\right). \end{aligned} \quad (3.19)$$

This is a classical analogue of the standard q -deformation of $sl(2)$ [32, 33].

In addition, there exists another set of non-local conserved charges,

$$\tilde{Q}^{\hat{\pm}} = -\frac{L}{2} \int_{-\infty}^{\infty} dx \, e^{-\alpha\chi(x)} n^{\hat{\pm}}(x). \quad (3.20)$$

These can be obtained by changing the sign of α in $Q^{\hat{\pm}}$ and hence Q^0 and $\tilde{Q}^{\hat{\pm}}$ also generate another q -deformed $sl(2)$ algebra,

$$\begin{aligned} \left\{ \tilde{Q}^{\hat{\pm}}, Q^0 \right\}_{\text{P}} &= \pm i \tilde{Q}^{\hat{\pm}}, \\ \left\{ \tilde{Q}^{\hat{+}}, \tilde{Q}^{\hat{-}} \right\}_{\text{P}} &= i \frac{L}{2\alpha} \sinh\left(\frac{2\alpha}{L} Q^0\right). \end{aligned} \quad (3.21)$$

In subsection 3.4, we will show that a (classical analogue of) quantum affine algebra is generated by Q^0 , $Q^{\hat{\pm}}$ and $\tilde{Q}^{\hat{\pm}}$ in the sense of Drinfeld's first realization [31].

3.3 Monodromy expansions and higher non-local charges

An infinite number of conserved charges are obtained by expanding the monodromy matrix $M(z)$ with respect to a complex parameter $u = e^{-z}$. Here we derive the conserved charges discussed in the previous subsection by expanding $M(z)$. According to the expansion, other higher non-local charges are also obtained.

Depending on the values of u , the following two expansions are possible:

$$\begin{aligned} \text{i)} \quad M(z) &= e^{\bar{q}_0} \exp \left[\sum_{n=1}^{\infty} u^n \bar{q}_n \right] \quad \text{for } |u| \ll 1, \\ \text{ii)} \quad M(z) &= e^{q_0} \exp \left[\sum_{n=1}^{\infty} u^{-n} q_n \right] \quad \text{for } |u| \gg 1. \end{aligned}$$

Let us consider each of the two expansions below.

Expansion i)

The monodromy matrix is expanded like

$$\begin{aligned} M(z) &= \text{P exp} \left[\int_{-\infty}^{\infty} dx U(t, x; z) \right] \\ &= e^{i \frac{2\alpha}{L} Q_{(0)}^0 T^0} \text{P exp} \left[2i\alpha \int_{-\infty}^{\infty} dx \left\{ (u + u^3) \left(T^{\hat{+}} e^{-\frac{\alpha}{L} Q_{(0)}^0} e^{\alpha x} n^{\hat{-}}(x) + T^{\hat{-}} e^{\frac{\alpha}{L} Q_{(0)}^0} e^{-\alpha x} n^{\hat{+}}(x) \right) \right. \right. \\ &\quad \left. \left. - u^2 T^0 n^0(x) + \mathcal{O}(u^4) \right\} \right] \\ &= e^{\bar{q}_0} \left[1 + u \bar{q}_1 + u^2 \left(\bar{q}_2 + \frac{1}{2} (\bar{q}_1)^2 \right) + u^3 \left(\bar{q}_3 + \frac{1}{2} (\bar{q}_2 \bar{q}_1 + \bar{q}_1 \bar{q}_2) + \frac{1}{6} (\bar{q}_1)^3 \right) + \mathcal{O}(u^4) \right]. \end{aligned}$$

where \bar{q}_i ($i = 0, 1, 2, 3, \dots$) are defined as

$$\begin{aligned} \bar{q}_0 &\equiv i \frac{2\alpha}{L} T^0 Q_{(0)}^0, \quad \bar{q}_1 \equiv -2i \frac{2\alpha}{L} \left(T^{\hat{+}} e^{-\frac{\alpha}{L} Q_{(0)}^0} Q_{(1)}^{\hat{-}} + T^{\hat{-}} e^{\frac{\alpha}{L} Q_{(0)}^0} \tilde{Q}_{(1)}^{\hat{+}} \right), \\ \bar{q}_2 &\equiv -2i \left(\frac{2\alpha}{L} \right)^2 T^0 \bar{Q}_{(2)}^0, \quad \bar{q}_3 \equiv 2i \left(\frac{2\alpha}{L} \right)^3 \left(T^{\hat{+}} e^{-\frac{\alpha}{L} Q_{(0)}^0} Q_{(3)}^{\hat{-}} + T^{\hat{-}} e^{\frac{\alpha}{L} Q_{(0)}^0} \tilde{Q}_{(3)}^{\hat{+}} \right), \\ &\vdots \end{aligned} \tag{3.22}$$

Then the conserved charges are given by

$$\begin{aligned} Q_{(0)}^0 &= -\frac{L}{2} \int_{-\infty}^{\infty} dx n^0(x), \\ Q_{(1)}^{\hat{-}} &= -\frac{L}{2} \int_{-\infty}^{\infty} dx e^{\alpha x} n^{\hat{-}}(x), \quad \tilde{Q}_{(1)}^{\hat{+}} = -\frac{L}{2} \int_{-\infty}^{\infty} dx e^{-\alpha x} n^{\hat{+}}(x), \\ \bar{Q}_{(2)}^0 &= \left(-\frac{L}{2} \right)^2 \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \epsilon(x-y) e^{\alpha x} n^{\hat{-}}(x) e^{-\alpha x} n^{\hat{+}}(y) - \frac{L}{2\alpha} Q_{(0)}^0, \\ Q_{(3)}^{\hat{-}} &= \left(-\frac{L}{2} \right)^3 \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz \frac{1}{2} \epsilon(x-y) \epsilon(x-z) e^{-\alpha x} n^{\hat{+}}(x) e^{\alpha x} n^{\hat{-}}(y) e^{\alpha x} n^{\hat{-}}(z) \end{aligned}$$

$$\begin{aligned}
& -\left(-\frac{L}{2}\right)^2 \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \frac{L}{2\alpha} \epsilon(x-y) e^{\alpha\chi} n^{\widehat{-}}(x) n^0(y) - \frac{1}{6} \widetilde{Q}_{(1)}^{\widehat{+}} (Q_{(1)}^{\widehat{-}})^2 + \left(\frac{L}{2\alpha}\right)^2 Q_{(1)}^{\widehat{-}}, \\
\widetilde{Q}_{(3)}^{\widehat{+}} &= \left(-\frac{L}{2}\right)^3 \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz \frac{1}{2} \epsilon(x-y) \epsilon(x-z) e^{\alpha\chi} n^{\widehat{-}}(x) e^{-\alpha\chi} n^{\widehat{+}}(y) e^{-\alpha\chi} n^{\widehat{+}}(z) \\
& + \left(-\frac{L}{2}\right)^2 \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \frac{L}{2\alpha} \epsilon(x-y) e^{-\alpha\chi} n^{\widehat{+}}(x) n^0(y) - \frac{1}{6} Q_{(1)}^{\widehat{-}} (\widetilde{Q}_{(1)}^{\widehat{+}})^2 + \left(\frac{L}{2\alpha}\right)^2 \widetilde{Q}_{(1)}^{\widehat{+}}, \\
& \vdots
\end{aligned} \tag{3.23}$$

Expansion ii)

The next is to consider the expansion of the monodromy matrix in the region ii). For this purpose, it is convenient to introduce a new parameter $\tilde{u} = 1/u$. Then the monodromy matrix is expanded in terms of \tilde{u} like

$$\begin{aligned}
M(z) &= \text{P exp} \left[\int_{-\infty}^{\infty} dx U(t, x; z) \right] \\
&= e^{-i\frac{2\alpha}{L} Q_{(0)}^0 T^0} \text{P exp} \left[2i\alpha \int_{-\infty}^{\infty} dx \left\{ -(\tilde{u} + \tilde{u}^3) \left(T^{\widehat{+}} e^{\frac{\alpha}{L} Q_{(0)}^0} e^{-\alpha\chi} n^{\widehat{-}}(x) + T^{\widehat{-}} e^{-\frac{\alpha}{L} Q_{(0)}^0} e^{\alpha\chi} n^{\widehat{+}}(x) \right) \right. \right. \\
&\quad \left. \left. + \tilde{u}^2 T^0 n^0(x) + \mathcal{O}(\tilde{u}^4) \right\} \right] \\
&= e^{q_0} \left[1 + \tilde{u} q_1 + \tilde{u}^2 \left(q_2 + \frac{1}{2} (q_1)^2 \right) + \tilde{u}^3 \left(q_3 + \frac{1}{2} (q_2 q_1 + q_1 q_2) + \frac{1}{6} (q_1)^3 \right) + \mathcal{O}(\tilde{u}^4) \right],
\end{aligned}$$

where q_i ($i = 0, 1, 2, 3, \dots$) are defined as

$$\begin{aligned}
q_0 &\equiv -i\frac{2\alpha}{L} T^0 Q_{(0)}^0, & q_1 &\equiv 2i\frac{2\alpha}{L} \left(T^{\widehat{+}} e^{\frac{\alpha}{L} Q_{(0)}^0} \widetilde{Q}_{(1)}^{\widehat{-}} + T^{\widehat{-}} e^{-\frac{\alpha}{L} Q_{(0)}^0} Q_{(1)}^{\widehat{+}} \right), \\
q_2 &\equiv 2i\left(\frac{2\alpha}{L}\right)^2 T^0 Q_{(2)}^0, & q_3 &\equiv -2i\left(\frac{2\alpha}{L}\right)^3 \left(T^{\widehat{+}} e^{\frac{\alpha}{L} Q_{(0)}^0} \widetilde{Q}_{(3)}^{\widehat{-}} + T^{\widehat{-}} e^{-\frac{\alpha}{L} Q_{(0)}^0} Q_{(3)}^{\widehat{+}} \right), \\
&\vdots
\end{aligned} \tag{3.24}$$

The conserved charges are given by

$$\begin{aligned}
Q_{(0)}^0 &= -\frac{L}{2} \int_{-\infty}^{\infty} dx n^0(x), \\
\widetilde{Q}_{(1)}^{\widehat{-}} &= -\frac{L}{2} \int_{-\infty}^{\infty} dx e^{-\alpha\chi} n^{\widehat{-}}(x), & Q_{(1)}^{\widehat{+}} &= -\frac{L}{2} \int_{-\infty}^{\infty} dx e^{\alpha\chi} n^{\widehat{+}}(x), \\
Q_{(2)}^0 &= \left(-\frac{L}{2}\right)^2 \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \epsilon(x-y) e^{\alpha\chi} n^{\widehat{+}}(x) e^{-\alpha\chi} n^{\widehat{-}}(y) - \frac{L}{2\alpha} Q_{(0)}^0, \\
\widetilde{Q}_{(3)}^{\widehat{-}} &= \left(-\frac{L}{2}\right)^3 \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz \frac{1}{2} \epsilon(x-y) \epsilon(x-z) e^{\alpha\chi} n^{\widehat{+}}(x) e^{-\alpha\chi} n^{\widehat{-}}(y) e^{-\alpha\chi} n^{\widehat{-}}(z) \\
&\quad + \left(-\frac{L}{2}\right)^2 \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \frac{L}{2\alpha} \epsilon(x-y) e^{-\alpha\chi} n^{\widehat{+}}(x) n^0(y) - \frac{1}{6} Q_{(1)}^{\widehat{+}} (\widetilde{Q}_{(1)}^{\widehat{-}})^2 + \left(\frac{L}{2\alpha}\right)^2 \widetilde{Q}_{(1)}^{\widehat{-}}, \\
Q_{(3)}^{\widehat{+}} &= \left(-\frac{L}{2}\right)^3 \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz \frac{1}{2} \epsilon(x-y) \epsilon(x-z) e^{-\alpha\chi} n^{\widehat{-}}(x) e^{\alpha\chi} n^{\widehat{+}}(y) e^{\alpha\chi} n^{\widehat{+}}(z) \\
&\quad - \left(-\frac{L}{2}\right)^2 \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \frac{L}{2\alpha} \epsilon(x-y) e^{\alpha\chi} n^{\widehat{+}}(x) n^0(y) - \frac{1}{6} \widetilde{Q}_{(1)}^{\widehat{-}} (Q_{(1)}^{\widehat{+}})^2 + \left(\frac{L}{2\alpha}\right)^2 Q_{(1)}^{\widehat{+}},
\end{aligned}$$

$$\vdots \quad (3.25)$$

In total, we have derived all of the conserved charges presented in the previous subsection, in addition to higher non-local charges.

It is a turn to clarify the algebraic structure of the conserved charges. Some examples of the Poisson brackets are listed below:

$$\begin{aligned} \left\{ Q_{(1)}^{\pm}, Q_{(1)}^{\mp} \right\}_{\text{P}} &= \pm i \frac{L}{2\alpha} \sinh\left(\frac{2\alpha}{L} Q_{(0)}^0\right), \quad \left\{ Q_{(1)}^{\pm}, Q_{(1)}^{\pm} \right\}_{\text{P}} = 0, \quad \left\{ Q_{(1)}^{\pm}, Q_{(0)}^0 \right\}_{\text{P}} = \pm i Q_{(1)}^{\pm}, \\ \left\{ \tilde{Q}_{(1)}^{\pm}, \tilde{Q}_{(1)}^{\mp} \right\}_{\text{P}} &= \pm i \frac{L}{2\alpha} \sinh\left(\frac{2\alpha}{L} Q_{(0)}^0\right), \quad \left\{ \tilde{Q}_{(1)}^{\pm}, \tilde{Q}_{(1)}^{\pm} \right\}_{\text{P}} = 0, \quad \left\{ \tilde{Q}_{(1)}^{\pm}, Q_{(0)}^0 \right\}_{\text{P}} = \pm i \tilde{Q}_{(1)}^{\pm}, \\ \left\{ Q_{(1)}^{\pm}, \tilde{Q}_{(1)}^{\pm} \right\}_{\text{P}} &= 0, \quad \left\{ Q_{(1)}^{\pm}, \tilde{Q}_{(1)}^{\mp} \right\}_{\text{P}} = -i \frac{2\alpha}{L} Q_{(2)}^0, \quad \left\{ Q_{(1)}^{\mp}, \tilde{Q}_{(1)}^{\mp} \right\}_{\text{P}} = i \frac{2\alpha}{L} \bar{Q}_{(2)}^0, \\ \left\{ Q_{(2)}^0, Q_{(1)}^{\pm} \right\}_{\text{P}} &= i \frac{2\alpha}{L} \left[Q_{(3)}^{\pm} + \frac{2}{3} \tilde{Q}_{(1)}^{\mp} (Q_{(1)}^{\pm})^2 \right], \\ \left\{ Q_{(2)}^0, \tilde{Q}_{(1)}^{\mp} \right\}_{\text{P}} &= -i \frac{2\alpha}{L} \left[\tilde{Q}_{(3)}^{\mp} + \frac{2}{3} Q_{(1)}^{\mp} (\tilde{Q}_{(1)}^{\mp})^2 \right], \\ \left\{ \bar{Q}_{(2)}^0, Q_{(1)}^{\mp} \right\}_{\text{P}} &= -i \frac{2\alpha}{L} \left[Q_{(3)}^{\mp} + \frac{2}{3} \tilde{Q}_{(1)}^{\mp} (Q_{(1)}^{\mp})^2 \right], \\ \left\{ \bar{Q}_{(2)}^0, \tilde{Q}_{(1)}^{\mp} \right\}_{\text{P}} &= i \frac{2\alpha}{L} \left[\tilde{Q}_{(3)}^{\mp} + \frac{2}{3} Q_{(1)}^{\mp} (\tilde{Q}_{(1)}^{\mp})^2 \right], \\ \left\{ Q_{(3)}^{\pm}, Q_{(1)}^{\pm} \right\}_{\text{P}} &= \frac{i}{3} \frac{2\alpha}{L} Q_{(2)}^0 (Q_{(1)}^{\pm})^2, \quad \left\{ \tilde{Q}_{(3)}^{\mp}, \tilde{Q}_{(1)}^{\mp} \right\}_{\text{P}} = -\frac{i}{3} \frac{2\alpha}{L} Q_{(2)}^0 (\tilde{Q}_{(1)}^{\mp})^2, \\ \left\{ Q_{(3)}^{\mp}, Q_{(1)}^{\mp} \right\}_{\text{P}} &= -\frac{i}{3} \frac{2\alpha}{L} \bar{Q}_{(2)}^0 (Q_{(1)}^{\mp})^2, \quad \left\{ \tilde{Q}_{(3)}^{\mp}, \tilde{Q}_{(1)}^{\mp} \right\}_{\text{P}} = \frac{i}{3} \frac{2\alpha}{L} \bar{Q}_{(2)}^0 (\tilde{Q}_{(1)}^{\mp})^2. \end{aligned}$$

3.4 Quantum affine algebra

Let us see the relation to a classical analogue of Drinfeld's first realization of quantum affine algebra [31]. It is convenient to rescale the charges $Q_{(0)}^0, Q_{(1)}^{\pm}$ and $\tilde{Q}_{(1)}^{\pm}$ as follows:

$$\begin{aligned} H_1 &\equiv 2Q_{(0)}^0, & H_0 &\equiv -2Q_{(0)}^0, \\ E_1 &\equiv \left(\frac{-2\alpha/L}{\sinh(\alpha/L)} \right)^{1/2} Q_{(1)}^{\pm}, & E_0 &\equiv \left(\frac{-2\alpha/L}{\sinh(\alpha/L)} \right)^{1/2} \tilde{Q}_{(1)}^{\mp}, \\ F_1 &\equiv \left(\frac{-2\alpha/L}{\sinh(\alpha/L)} \right)^{1/2} Q_{(1)}^{\mp}, & F_0 &\equiv \left(\frac{-2\alpha/L}{\sinh(\alpha/L)} \right)^{1/2} \tilde{Q}_{(1)}^{\mp}. \end{aligned} \quad (3.26)$$

Then the Poisson brackets are evaluated as

$$\begin{aligned} i\{H_i, H_j\}_{\text{P}} &= 0 \quad (i, j = 1, 0), \\ i\{H_i, E_j\}_{\text{P}} &= A_{ij} E_j, \quad i\{H_i, F_j\}_{\text{P}} = -A_{ij} F_j, \\ i\{E_i, F_j\}_{\text{P}} &= \delta_{ij} \frac{q^{H_i} - q^{-H_i}}{q - q^{-1}}, \end{aligned} \quad (3.27)$$

where the generalized Cartan matrix A_{ij} is given by

$$A_{ij} = (\alpha_i, \alpha_j) = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \quad \text{with} \quad \alpha_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \alpha_0 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad (3.28)$$

and a q -deformation parameter is defined as

$$q \equiv e^{\alpha/L}. \quad (3.29)$$

The remaining task is to check a classical analogue of q -Serre relations, which are deduced by introducing the classical q -Poisson bracket [22],

$$\{J^A, J^B\}_{qP} \equiv \{J^A, J^B\}_P + \frac{\alpha}{L}(\beta_A, \beta_B)J^B J^A. \quad (3.30)$$

Here β_A are the associated root vectors. Now J^A and J^B are c -number and commutative, hence the ordering in the second term is irrelevant.

By a direct computations with the Poisson brackets computed in subsection 3.3, the classical q -Serre relations are evaluated as

$$\{E_i, \{E_i, \{E_i, E_j\}_{qP}\}_{qP}\}_{qP} = \{F_i, \{F_i, \{F_i, F_j\}_{qP}\}_{qP}\}_{qP} = 0 \quad \text{for } |i - j| = 1. \quad (3.31)$$

The q -Serre relations can also be rewritten in terms of $Q_{(1)}^{\pm}$ and $\tilde{Q}_{(1)}^{\pm}$ like

$$\{Q_{(1)}^{\pm}, \{Q_{(1)}^{\pm}, \{Q_{(1)}^{\pm}, \tilde{Q}_{(1)}^{\mp}\}_{qP}\}_{qP}\}_{qP} = 0, \quad (3.32)$$

$$\{\tilde{Q}_{(1)}^{\mp}, \{\tilde{Q}_{(1)}^{\mp}, \{\tilde{Q}_{(1)}^{\mp}, Q_{(1)}^{\pm}\}_{qP}\}_{qP}\}_{qP} = 0. \quad (3.33)$$

Thus we have shown the classical analogue of quantum affine algebra in the sense of Drinfeld's first realization [31]. This algebra can be interpreted as the remnant of the relativistic case [22] after taking the fast-moving string limit, while the left Yangians are not realized any more.

3.5 The classical r -matrix

Finally, let us comment on the classical r -matrix. One can read off the classical r -matrix from the following Poisson bracket of $U(t, x; z)$,

$$\{U(t, x; z) \otimes U(t, y; w)\}_P = [r(z - w), U(t, x; z) \otimes 1 + 1 \otimes U(t, y; w)] \delta(x - y). \quad (3.34)$$

This bracket has been evaluated by using the Poisson brackets (3.18). Note that this bracket does not contain non-ultra local terms in comparison to the relativistic case like principal chiral models. Hence there is no difficulty to read off the classical r -matrix. This is an advantage to consider the fast-moving string limit.

The resulting classical r -matrix is of trigonometric type [57],

$$r(z - w) = \frac{2}{L} \frac{\alpha}{\sinh(z - w)} \left[-\cosh(z - w) T^0 \otimes T^0 + T^1 \otimes T^1 + T^2 \otimes T^2 \right], \quad (3.35)$$

and satisfies the classical Yang-Baxter equation,

$$[r_{12}(z - w), r_{13}(z - u)] + [r_{12}(z - w), r_{23}(w - u)] + [r_{13}(z - u), r_{23}(w - u)] = 0. \quad (3.36)$$

Thus it ensures that the system is classically integrable.

4 Integrability of null-like warped $SL(2)$ LLSM

In this section, we consider a *null-like warped $SL(2)$ LLSM*. This system is derived as a pp-wave like limit of the time-like warped $SL(2)$ LLSM, as shown in detail in Appendix D. The resulting system also coincides with a fast-moving string limit of $Sch_3 \times S^1$ subsector of the Jordanian deformed $AdS_5 \times S^5$ superstring action [41, 42]. The associated infinite-dimensional symmetries are also discussed by performing non-local gauge transformations which correspond to Jordanian twists.

4.1 The classical action and Lax pair

The classical action of the null-like warped $SL(2)$ LLSM is given by [54],

$$S = \frac{L}{2} \int_{-\infty}^{\infty} dt dx \left[-\frac{C}{2} (\cosh \rho + \sin \psi \sinh \rho)^2 + (\cosh \rho - 1) \partial_t \psi - \frac{\lambda}{16\pi^2 L^2} \left[(\partial_x \rho)^2 + \sinh^2 \rho (\partial_x \psi)^2 \right] \right]. \quad (4.1)$$

This system is derived as a pp-wave like limit of the time-like warped $SL(2)$ LLSM, as shown in detail in Appendix D. The resulting system also coincides with a fast-moving string limit of $Sch_3 \times S^1$ subsector of the Jordanian deformed $AdS_5 \times S^5$ superstring action [41, 42].

It is convenient to introduce a vector notation n^a ,

$$n^0 = -\cosh \rho, \quad n^1 = \sinh \rho \sin \psi, \quad n^2 = \sinh \rho \cos \psi. \quad (4.2)$$

Then the classical equations of motion are written as

$$\begin{aligned} \partial_t n^0 &= \frac{\lambda}{8\pi^2 L^2} (n^1 \partial_x^2 n^2 - n^2 \partial_x^2 n^1) - C (n^0 - n^1) n^2, \\ \partial_t n^1 &= \frac{\lambda}{8\pi^2 L^2} (n^0 \partial_x^2 n^2 - n^2 \partial_x^2 n^0) - C (n^0 - n^1) n^2, \\ \partial_t n^2 &= \frac{\lambda}{8\pi^2 L^2} (-n^0 \partial_x^2 n^1 + n^1 \partial_x^2 n^0) - C (n^0 - n^1)^2. \end{aligned} \quad (4.3)$$

These are summarized to a simpler form

$$\partial_t n_a = \varepsilon_{abc} n^b \left(\frac{\lambda}{8\pi^2 L^2} \partial_x^2 n^c + \mathcal{J}_d^c n^d \right) \quad (a = 0, 1, 2), \quad (4.4)$$

with the anisotropic matrix \mathcal{J}

$$\mathcal{J}_b^a = \begin{pmatrix} j + C & -C & 0 \\ C & j - C & 0 \\ 0 & 0 & j \end{pmatrix} \quad (j : \text{an arbitrary const.}). \quad (4.5)$$

The equations (4.4) describe null-like deformed Landau-Lifshitz equations.

Lax pair. First of all, it is helpful to introduce the light-cone expressions as

$$n^\pm \equiv \frac{n^0 \pm n^1}{\sqrt{2}}. \quad (4.6)$$

Then the Lax pair of the null-like warped $SL(2)$ LLSM is given by

$$\begin{aligned} U(t, x; z) &= \frac{\alpha}{z} \left[- \left(n^+ - \frac{Cz^2}{2} n^- \right) T^- + n^2 T^2 - n^- T^+ \right], \\ V(t, x; z) &= \frac{\beta}{z} \left[- \left((n^2 \partial_x n^+ - n^+ \partial_x n^2) - \frac{Cz^2}{2} (n^- \partial_x n^2 - n^2 \partial_x n^-) \right) T^- \right. \\ &\quad \left. + (n^- \partial_x n^+ - n^+ \partial_x n^-) T^2 - (n^- \partial_x n^2 - n^2 \partial_x n^-) T^+ \right] \\ &\quad + \frac{\alpha\beta}{z^2} \left[- \left(n^+ + \frac{Cz^2}{2} n^- \right) T^- + n^2 T^2 - n^- T^+ \right], \end{aligned} \quad (4.7)$$

where a spectral parameter $z \in \mathbb{C}$ and new parameters have been introduced as

$$\alpha \equiv \frac{4\pi L}{\sqrt{\lambda}}, \quad \beta \equiv -\frac{\sqrt{\lambda}}{2\pi L}. \quad (4.8)$$

The Lax pair (4.7) can be derived from the Lax pair of the time-like warped $SL(2)$ LLSM by taking a scaling limit, as shown in Appendix D.

The classical equations of motion (4.3) are reproduced from the commutation relation

$$\left[\partial_t - V(t, x; z), \partial_x - U(t, x; z) \right] = 0, \quad (4.9)$$

and one can check that the Lax pair (4.7) works well.

4.2 q -deformed Poincaré algebras

Let us consider the symmetry of the null-like warped $SL(2)$ LLSM. Due to the deformation, the original $SL(2)$ symmetry is broken to $U(1)$. As in the previous cases, the broken components are still realized as non-local symmetries.

The boundary condition is sensitive to the argument on the non-local symmetries. In the present case, it is supposed that n^- and n^2 vanish at the spatial infinities. The detail of the boundary condition is described in Appendix C.

The unbroken $U(1)$ charge is constructed as

$$Q^- = -\frac{L}{2} \int_{-\infty}^{\infty} dx n^-(x). \quad (4.10)$$

For the broken components $+$ and 2 , one can find non-local conserved charges

$$Q^+ = -\frac{L}{2} \int_{-\infty}^{\infty} dx e^{\sqrt{C}\alpha\chi(x)} n^+(x), \quad Q^2 = -\frac{L}{2} \int_{-\infty}^{\infty} dx e^{\sqrt{C}\alpha\chi(x)} n^2(x). \quad (4.11)$$

Here $\chi(x)$ is a non-local field defined as

$$\chi(x) \equiv \frac{1}{2} \int_{-\infty}^{\infty} dy \epsilon(x-y) n^-(y). \quad (4.12)$$

The non-local field satisfies the following relations,

$$\partial_x \chi = n^-, \quad \partial_t \chi = -\frac{\lambda}{8\pi^2 L^2} (n^- \partial_x n^2 - n^2 \partial_x n^-). \quad (4.13)$$

These are useful to show the conservation laws of Q^+ and Q^2 .

The next task is to compute the Poisson brackets of Q^2 and Q^\pm . The Poisson brackets of the dynamical variables n^a ($a = 2, \pm$) are

$$\{n^a(x), n^b(y)\}_P = -\frac{2}{L} \varepsilon^{ab}{}_c n^c(x) \delta(x-y). \quad (4.14)$$

With (4.14), the Poisson brackets of the charges are evaluated as

$$\begin{aligned} \{Q^+, Q^-\}_P &= -Q^2, \\ \{Q^+, Q^2\}_P &= -Q^+ \cosh\left(\frac{\sqrt{C}\alpha}{L} Q^-\right), \\ \{Q^-, Q^2\}_P &= \frac{L}{\sqrt{C}\alpha} \sinh\left(\frac{\sqrt{C}\alpha}{L} Q^-\right). \end{aligned} \quad (4.15)$$

This is a classical analogue of a non-standard q -deformation of $sl(2)$, where the deformation parameter q is defined as

$$q \equiv e^{\frac{\sqrt{C}\alpha}{L}}.$$

The resulting Poisson algebra (4.15) is isomorphic to a q -deformed Poincaré algebra [58, 59] with an appropriate rescaling the charges.

In addition, there exists another set of non-local conserved charges,

$$\tilde{Q}^+ = -\frac{L}{2} \int_{-\infty}^{\infty} dx e^{-\sqrt{C}\alpha\chi(x)} n^+(x), \quad \tilde{Q}^2 = -\frac{L}{2} \int_{-\infty}^{\infty} dx e^{-\sqrt{C}\alpha\chi(x)} n^2(x). \quad (4.16)$$

These are obtained by flipping the sign of \sqrt{C} in Q^+ and Q^2 . By construction, the charges Q^-, \tilde{Q}^+ and \tilde{Q}^2 also generate another q -deformed Poincaré algebra.

Note that the mixed Poisson brackets like $\{Q^a, \tilde{Q}^b\}_P$ should be taken into account. Then the algebra is extended to an infinite-dimensional symmetry referred to as the *exotic* symmetry in [24]. It is straightforward to reproduce this infinite-dimensional algebra, but we will not do that here. Instead, we will derive Yangians by undoing Jordanian twists in the next subsection.

It would be interesting to argue the corresponding classical r -matrix. From the following Poisson bracket

$$\{U(t, x; z) \otimes U(t, y; w)\}_P = [r(z-w), U(t, x; z) \otimes 1 + 1 \otimes U(t, y; w)] \delta(x-y). \quad (4.17)$$

the classical r -matrix is easily obtained. There is no difficulty of non-ultra local terms again. The resulting r -matrix is deformed like

$$r(z-w) = \frac{2}{L} \left[\frac{\alpha}{z-w} \gamma_{ab} T^a \otimes T^b + \frac{C\alpha}{2} (z-w) T^- \otimes T^- \right], \quad (4.18)$$

but it still satisfies the classical Yang-Baxter equation (3.36). The r -matrix of this type is discussed in [39].

4.3 Jordanian twists and $sl(2)$ Yangians

Let us consider infinite-dimensional symmetries by performing non-local gauge transformations. It has been shown in [25] that the null-like deformation may be interpreted as Jordanian twists in the relativistic case. In fact, this interpretation is still applicable in the present non-relativistic case. That is, the structure of Jordanian twists remains even after taking the fast-moving string limit.

First of all, by following [25], let us derive isotropic Lax pairs $(\mathcal{U}^{(\pm)}(z), \mathcal{V}^{(\pm)}(z))$. These are obtained from the anisotropic Lax pair (4.7) by performing non-local gauge transformations. The derivation of the isotropic Lax pairs are described in Appendix E.

The resulting isotropic Lax pairs $(\mathcal{U}^{(\pm)}(z), \mathcal{V}^{(\pm)}(z))$ are given by

$$\begin{aligned} \mathcal{U}^{(+)}(z) &= \frac{\alpha}{z} \left[-\mathcal{N}^- T^+ + \mathcal{N}^2 T^2 - \mathcal{N}^+ T^- \right], \\ \mathcal{V}^{(+)}(z) &= \frac{\beta}{z} \left[-(\mathcal{N}^- \partial_x \mathcal{N}^2 - \mathcal{N}^2 \partial_x \mathcal{N}^-) T^+ + (\mathcal{N}^- \partial_x \mathcal{N}^+ - \mathcal{N}^+ \partial_x \mathcal{N}^-) T^2 \right. \\ &\quad \left. - (\mathcal{N}^2 \partial_x \mathcal{N}^+ - \mathcal{N}^+ \partial_x \mathcal{N}^2) T^- \right] + \frac{\alpha\beta}{z^2} \left[-\mathcal{N}^- T^+ + \mathcal{N}^2 T^2 - \mathcal{N}^+ T^- \right], \\ \mathcal{U}^{(-)}(z) &= \frac{\alpha}{z} \left[-\tilde{\mathcal{N}}^- T^+ + \tilde{\mathcal{N}}^2 T^2 - \tilde{\mathcal{N}}^+ T^- \right], \\ \mathcal{V}^{(-)}(z) &= \frac{\beta}{z} \left[-(\tilde{\mathcal{N}}^- \partial_x \tilde{\mathcal{N}}^2 - \tilde{\mathcal{N}}^2 \partial_x \tilde{\mathcal{N}}^-) T^+ + (\tilde{\mathcal{N}}^- \partial_x \tilde{\mathcal{N}}^+ - \tilde{\mathcal{N}}^+ \partial_x \tilde{\mathcal{N}}^-) T^2 \right. \\ &\quad \left. - (\tilde{\mathcal{N}}^2 \partial_x \tilde{\mathcal{N}}^+ - \tilde{\mathcal{N}}^+ \partial_x \tilde{\mathcal{N}}^2) T^- \right] + \frac{\alpha\beta}{z^2} \left[-\tilde{\mathcal{N}}^- T^+ + \tilde{\mathcal{N}}^2 T^2 - \tilde{\mathcal{N}}^+ T^- \right]. \end{aligned} \quad (4.19)$$

Here \mathcal{N}^a and $\tilde{\mathcal{N}}^a$ are the components of non-local unit-vectors defined in Appendix E. Note that there are two kinds of Lax pair according to the choice of the twists (non-local gauge transformations). The analysis to be performed below is almost irrelevant to the choice. Hence we will concentrate on the (+) superscript hereafter.

Now there is a great advantage because one can follow the standard prescription to consider the classical integrable structure. With a general prescription, the monodromy matrix is constructed. Then, by expanding the monodromy matrix, an infinite number of conserved charges are obtained. Here we shall list the first three charges:

$$\begin{aligned} \mathcal{Q}_{(0)}^a &= -\frac{L}{2} \int_{-\infty}^{\infty} dx \mathcal{N}^a(x), \\ \mathcal{Q}_{(1)}^a &= \left(-\frac{L}{2} \right)^2 \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \frac{1}{4} \epsilon(x-y) \epsilon_{bc}^a \mathcal{N}^b(x) \mathcal{N}^c(y), \end{aligned}$$

$$\mathcal{Q}_{(2)}^a = \left(-\frac{L}{2}\right)^3 \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz \frac{1}{12} \epsilon(x-y) \epsilon(x-z) \gamma_{bc} \times \left[\mathcal{N}^a(x) \mathcal{N}^b(y) \mathcal{N}^c(z) - \mathcal{N}^b(x) \mathcal{N}^a(y) \mathcal{N}^c(z) \right]. \quad (4.20)$$

Note that these are quite similar to Yangian generators constructed from a conserved local current satisfying the flatness condition in the symmetric coset case. However, in the present case, the components of \mathcal{N}^a are non-local. Hence it is necessary to check the Poisson brackets of them concretely.

The Poisson brackets of \mathcal{N}^a are given by

$$\left\{ \mathcal{N}^a(x), \mathcal{N}^b(y) \right\}_{\text{P}} = -\frac{2}{L} \varepsilon^{ab}{}_c \mathcal{N}^c(x) \delta(x-y) \quad (a = 2, \pm). \quad (4.21)$$

With (4.21), the Poisson brackets of $\mathcal{Q}_{(0)}^a$ and $\mathcal{Q}_{(1)}^a$ are computed as

$$\begin{aligned} \left\{ \mathcal{Q}_{(0)}^a, \mathcal{Q}_{(0)}^b \right\}_{\text{P}} &= \varepsilon^{ab}{}_c \mathcal{Q}_{(0)}^c, & \left\{ \mathcal{Q}_{(1)}^a, \mathcal{Q}_{(0)}^b \right\}_{\text{P}} &= \varepsilon^{ab}{}_c \mathcal{Q}_{(1)}^c, \\ \left\{ \mathcal{Q}_{(1)}^a, \mathcal{Q}_{(1)}^b \right\}_{\text{P}} &= \varepsilon^{ab}{}_c \left[\mathcal{Q}_{(2)}^c - \frac{1}{12} \gamma_{ab} \mathcal{Q}_{(0)}^a \mathcal{Q}_{(0)}^b \mathcal{Q}_{(0)}^c \right]. \end{aligned} \quad (4.22)$$

These are the defining relations of Yangian in the sense of Drinfeld's first realization [31]. But the check of the defining relations has not been completed yet. The remaining task is to check the Serre relations. Indeed, one can show the following relations,

$$\begin{aligned} &\left\{ \left\{ \mathcal{Q}_{(1)}^+, \mathcal{Q}_{(1)}^- \right\}_{\text{P}}, \mathcal{Q}_{(1)}^2 \right\}_{\text{P}} = \frac{1}{4} \left(\mathcal{Q}_{(1)}^+ \mathcal{Q}_{(0)}^- - \mathcal{Q}_{(0)}^+ \mathcal{Q}_{(1)}^- \right) \mathcal{Q}_{(0)}^2, \\ &\left\{ \left\{ \mathcal{Q}_{(1)}^2, \mathcal{Q}_{(1)}^{\pm} \right\}_{\text{P}}, \mathcal{Q}_{(1)}^{\pm} \right\}_{\text{P}} = \frac{1}{4} \left(\mathcal{Q}_{(1)}^2 \mathcal{Q}_{(0)}^{\pm} - \mathcal{Q}_{(0)}^2 \mathcal{Q}_{(1)}^{\pm} \right) \mathcal{Q}_{(0)}^{\pm}, \\ &\left\{ \left\{ \mathcal{Q}_{(1)}^+, \mathcal{Q}_{(1)}^- \right\}_{\text{P}}, \mathcal{Q}_{(1)}^{\pm} \right\}_{\text{P}} \mp \left\{ \left\{ \mathcal{Q}_{(1)}^2, \mathcal{Q}_{(1)}^{\pm} \right\}_{\text{P}}, \mathcal{Q}_{(1)}^2 \right\}_{\text{P}} \\ &= \frac{1}{4} \left(\mathcal{Q}_{(1)}^+ \mathcal{Q}_{(0)}^- - \mathcal{Q}_{(0)}^+ \mathcal{Q}_{(1)}^- \right) \mathcal{Q}_{(0)}^{\pm} \mp \frac{1}{4} \left(\mathcal{Q}_{(1)}^2 \mathcal{Q}_{(0)}^{\pm} - \mathcal{Q}_{(0)}^2 \mathcal{Q}_{(1)}^{\pm} \right) \mathcal{Q}_{(0)}^2, \end{aligned} \quad (4.23)$$

and the Serre relations are also satisfied. Note again that there is no non-ultra local term. Thus the Yangian algebra $\mathcal{Y}(sl(2))$ is generated in a well-defined manner.

Starting from the $(-)$ superscript starting from $\mathcal{U}^{(-)}(z)$ and $\mathcal{V}^{(-)}(z)$, one can derive the same result. That is, the same Poisson brackets are derived, up to the replacement of $\mathcal{Q}_{(n)}^a$ by $\tilde{\mathcal{Q}}_{(n)}^a$, while $\tilde{\mathcal{Q}}_{(n)}^a$ are not identical with $\mathcal{Q}_{(n)}^a$.

Finally let us comment on the classical r -matrix. From the Poisson brackets

$$\begin{aligned} &\left\{ \mathcal{U}^{(\pm)}(t, x; z) \otimes \mathcal{U}^{(\pm)}(t, y; w) \right\}_{\text{P}} \\ &= \left[r^{(\pm)}(z-w), \mathcal{U}^{(\pm)}((t, x; z) \otimes 1 + 1 \otimes \mathcal{U}^{(\pm)}(t, y; w)) \right] \delta(x-y), \end{aligned}$$

one can read off the classical r -matrices. These are Yang's r -matrix of rational type,

$$r^{(\pm)}(z-w) = \frac{2}{L} \frac{\alpha}{z-w} \gamma_{ab} T^a \otimes T^b, \quad (4.24)$$

and satisfy the classical Yang-Baxter equation (3.36). Note that it is independent of C .

4.4 The relation between q -Poincaré algebras and Yangians

It is worth listing the relation between the exotic symmetry and the Yangians.

For simplicity, let us concentrate on one of the Yangians generated by Q^\pm and Q^2 . The similar argument holds also for the other Yangian. The Yangian charges can be represented in terms of the q -deformed Poincaré charges Q^a and \tilde{Q}^a .

The level-zero Yangian charges are expressed as

$$\begin{aligned} \mathcal{Q}_{(0)}^+ &= e^{\frac{\sqrt{C}\alpha}{L}Q^-} Q^+ + \frac{\sqrt{C}\alpha}{L} \left(e^{\frac{\sqrt{C}\alpha}{L}Q^-} Q^2 \right)^2, \\ \mathcal{Q}_{(0)}^2 &= e^{\frac{\sqrt{C}\alpha}{L}Q^-} Q^2, \\ \mathcal{Q}_{(0)}^- &= \frac{1}{2\frac{\sqrt{C}\alpha}{L}} \left(1 - e^{-2\frac{\sqrt{C}\alpha}{L}Q^-} \right). \end{aligned} \quad (4.25)$$

Then the level-one charges are

$$\begin{aligned} \mathcal{Q}_{(1)}^+ &= \frac{1}{4} \left[\frac{1}{\frac{\sqrt{C}\alpha}{L}} \left\{ \left\{ \tilde{Q}^2, Q^+ \right\}_P, Q^+ \right\}_P + \left\{ \tilde{Q}^2, Q^+ \right\}_P Q^2 \right. \\ &\quad \left. - \left(\sinh\left(\frac{\sqrt{C}\alpha}{L}Q^-\right) \tilde{Q}^2 + \cosh\left(\frac{\sqrt{C}\alpha}{L}Q^-\right) Q^2 \right) Q^+ \right], \\ \mathcal{Q}_{(1)}^2 &= \frac{1}{4\frac{\sqrt{C}\alpha}{L}} \left(\left\{ \tilde{Q}^2, Q^+ \right\}_P - \cosh\left(\frac{\sqrt{C}\alpha}{L}Q^-\right) Q^+ \right) + \frac{1}{4} Q^2 \left(\tilde{Q}^2 - Q^2 \right), \\ \mathcal{Q}_{(1)}^- &= \frac{1}{4\frac{\sqrt{C}\alpha}{L}} \left(\tilde{Q}^2 - Q^2 \right). \end{aligned} \quad (4.26)$$

These relations are the same as the ones in the relativistic case [25]. That is, this structure survives the fast-moving string limit.

4.5 A possible relation between gravitational solutions

Finally, we should comment on a possible relation between the q -deformed $\text{AdS}_5 \times S^5$ and a Jordanian deformation of $\text{AdS}_5 \times S^5$ argued in [42].

The null-like warped $SL(2)$ LLSM is obtained as a pp-wave like limit of the time-like warped one, as explained in Appendix D. The identical null-like warped $SL(2)$ LLSM is also derived from a string sigma model on $\text{Sch}_3 \times S^1$. Then the geometry of $\text{Sch}_3 \times S^1$ is contained as a subspace of a Jordanian deformed $\text{AdS}_5 \times S^5$ [42].

Thus one may expect a relation between the q -deformed $\text{AdS}_5 \times S^5$ and the Jordanian deformed $\text{AdS}_5 \times S^5$. Indeed, this is the case. As explained in detail in [41], the r -matrix that leads to the Jordanian deformed solution [42] is constructed by performing a Jordanian twist of the r -matrix of Drinfeld-Jimbo type [34, 35]. Hence this observation suggests that the Jordanian twist at the r -matrix level corresponds to the pp-wave like limit at the geometry level, as depicted in Figure 1. It would be an interesting issue to make this correspondence more precise.

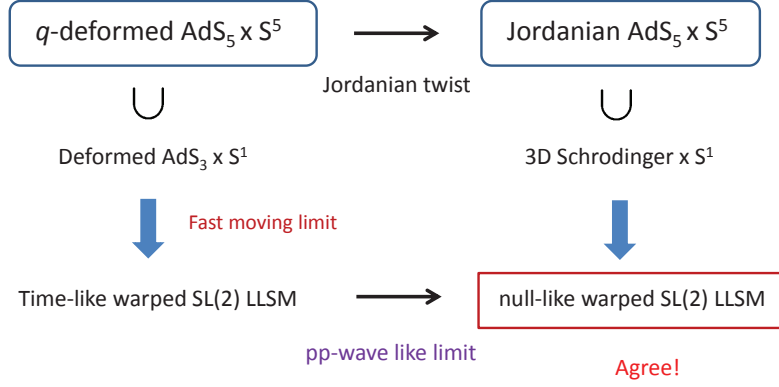


Figure 1. A possible relation between a Jordanian twist and a pp-wave like limit.

5 Conclusion and Discussion

In this paper, we have derived anisotropic LLSMs from bosonic subsectors of the q -deformed $\text{AdS}_5 \times S^5$ superstring action by taking fast-moving string limits. Then we have investigated the classical integrability of the LLSMs from the viewpoint of infinite-dimensional symmetries.

Concretely speaking, we have considered the subsectors, 1) deformed $\text{AdS}_3 \times S^1$ and 2) $R \times$ deformed S^3 . By taking fast-moving string limits, a time-like warped $SL(2)$ LLSM and a squashed S^3 LLSM have been derived for the case 1) and the case 2), respectively. It is remarkable that the resulting LLSMs coincide precisely with the ones obtained from a time-like warped AdS space and a squashed S^3 [54].

Then infinite-dimensional symmetries have been revealed under an appropriate boundary condition. In the case 1), a quantum affine algebra $U_q(\widehat{sl(2)})$ has been shown explicitly by computing the Poisson brackets of conserved non-local charges. In the case 2), a quantum affine algebra $U_q(\widehat{su(2)})$ is realized. It should be noted that non-ultra local terms do not appear in computing the Poisson algebra and hence there is no ambiguity in studying the classical integrable structure, in comparison to principal chiral models.

For the case 1), a pp-wave like limit has been applied. The resulting system coincides with a null-like warped $SL(2)$ LLSM obtained as a fast-moving string limit of a string sigma model on $\text{Sch}_3 \times S^1$ [54]. As a result, a couple of Yangians $\mathcal{Y}(sl(2))$ have been revealed by performing non-local gauge transformations which correspond to undoing Jordanian twists. In addition, we have argued a possible relation between the q -deformed $\text{AdS}_5 \times S^5$ and Jordanian deformed $\text{AdS}_5 \times S^5$.

It is interesting to consider some generalizations of our result. There would be various directions. The first is to consider a long-range generalization. By following the works [60–62], it would be possible to argue a long-range generalization of our result (For example, for a long-range generalization of the XXZ model see [63]). The next is to consider larger subsectors, or directly the full sector (For a supersymmetric generalization of the

undeformed case, see [64, 65]). It would also be nice to consider the LLSMs obtained here at the quantum level, for example, by following [66–68].

There are many open problems concerning anisotropic LLSMs. We hope that our result would open up a new arena to study q -deformations of the $\text{AdS}_5 \times \text{S}^5$ superstring.

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A Our convention of generators

We shall summarize here our conventions of the $sl(2)$ and $su(2)$ generators.

A.1 The $sl(2)$ generators

The $sl(2)$ generators T^a 's are defined as

$$T^0 \equiv \frac{i}{2}\sigma_2, \quad T^1 \equiv \frac{1}{2}\sigma_1, \quad T^2 \equiv \frac{1}{2}\sigma_3, \quad (\text{A.1})$$

where σ_i ($i = 1, 2, 3$) are the standard Pauli matrices. Then the commutation relation and the normalization are

$$[T^a, T^b] = \varepsilon^{ab}{}_c T^c, \quad \text{Tr} (T^a T^b) = \frac{1}{2} \gamma^{ab}, \quad (\text{A.2})$$

with a totally anti-symmetric tensor $\varepsilon^{ab}{}_c$ normalized as $\varepsilon^{01}{}_2 = +1$. The components of γ^{ab} are given by

$$-\gamma^{00} = \gamma^{11} = \gamma^{22} = +1, \quad \gamma^{ab} = 0 \quad (a \neq b). \quad (\text{A.3})$$

The $sl(2)$ indices are raised and lowered by γ^{ab} and its inverse, respectively.

It is convenient to introduce T^\pm defined as

$$T^\pm \equiv \frac{1}{\sqrt{2}} (T^0 \pm T^1), \quad (\text{A.4})$$

where the components of ε^{abc} and γ^{ab} are

$$\varepsilon^{-+2} = +1, \quad -\gamma^{-+} = -\gamma^{+-} = \gamma^{22} = +1.$$

It is also helpful to introduce other linear combinations $T^{\hat{\pm}}$ defined as

$$T^{\hat{\pm}} \equiv \frac{1}{\sqrt{2}} (T^1 \pm iT^2), \quad (\text{A.5})$$

The commutation relation and the normalization are

$$[T^a, T^b] = i \varepsilon^{ab}{}_c T^c, \quad \text{Tr} (T^a T^b) = \frac{1}{2} \gamma^{ab}, \quad (\text{A.6})$$

where the components of ε^{abc} and γ^{ab} are expressed as

$$\varepsilon^{0\hat{-}\hat{+}} = +1, \quad -\gamma^{00} = \gamma^{\hat{-}\hat{+}} = \gamma^{\hat{+}\hat{-}} = +1.$$

A.2 The $su(2)$ generators

The $su(2)$ generators T^a 's are defined as

$$T^1 \equiv \frac{1}{2i}\sigma_1, \quad T^2 \equiv \frac{1}{2i}\sigma_2, \quad T^3 \equiv \frac{1}{2i}\sigma_3, \quad (\text{A.7})$$

where σ_i ($i = 1, 2, 3$) are the standard Pauli matrices. Then the commutation relation and the normalization are given by

$$[T^a, T^b] = \varepsilon^{ab}{}_c T^c, \quad \text{Tr} (T^a T^b) = \frac{1}{2} \delta^{ab}, \quad (\text{A.8})$$

with a totally anti-symmetric tensor $\varepsilon^{ab}{}_c$ normalized as $\varepsilon^{12}{}_3 = +1$. The $su(2)$ indices are raised and lowered by δ^{ab} and its inverse, respectively.

It is convenient to introduce T^\pm defined as

$$T^\pm \equiv \frac{1}{\sqrt{2}} (T^1 \pm iT^2), \quad (\text{A.9})$$

where the components of ε^{abc} and δ^{ab} are given by

$$\varepsilon^{-+3} = +1, \quad \delta^{33} = \delta^{+-} = \delta^{-+} = +1.$$

B Integrability of squashed S^3 LLSM

Here we consider the classical integrability of the squashed S^3 LLSM and the associated infinite-dimensional symmetry.

B.1 The classical action and Lax pair

The classical action of the squashed S^3 LLSM is given by

$$S = \frac{L}{2} \int_{-\infty}^{\infty} dt dx \left[C \cos^2 \theta - \cosh \theta \partial_t \phi - \frac{\lambda}{16\pi^2 L^2} \left[(\partial_x \theta)^2 + \sin^2 \theta (\partial_x \phi)^2 \right] \right]. \quad (\text{B.1})$$

When $C = 0$, the isotropic S^3 LLSM is reproduced.

It is convenient to introduce a vector representation with n^a . The components are

$$n^1 = \sin \theta \sin \phi, \quad n^2 = \sin \theta \cos \phi, \quad n^3 = \cos \theta, \quad (\text{B.2})$$

and satisfy a constraint condition,

$$(n^1)^2 + (n^2)^2 + (n^3)^2 = 1. \quad (\text{B.3})$$

Then the classical equations of motion are rewritten as

$$\partial_t n^1 = \frac{\lambda}{8\pi^2 L^2} (n^2 \partial_x^2 n^3 - n^3 \partial_x^2 n^2) + 2C n^2 n^3,$$

$$\begin{aligned}\partial_t n^2 &= \frac{\lambda}{8\pi^2 L^2} (n^3 \partial_x^2 n^1 - n^1 \partial_x^2 n^3) - 2C n^1 n^3, \\ \partial_t n^3 &= \frac{\lambda}{8\pi^2 L^2} (n^1 \partial_x^2 n^2 - n^2 \partial_x^2 n^1) .\end{aligned}\tag{B.4}$$

These are equivalent to the Landau-Lifshitz equations¹

$$\partial_t n_a = \varepsilon_{abc} n^b \left(\frac{\lambda}{8\pi^2 L^2} \partial_x^2 n^c + \mathcal{J}_d n^d \right) \quad (a = 1, 2, 3),\tag{B.5}$$

with an anisotropic matrix \mathcal{J}

$$\mathcal{J}_b^a = \text{diag}(j, j, j + 2C) \quad (j : \text{an arbitrary const.}).\tag{B.6}$$

Lax pair and monodromy matrix

Let us introduce the Lax pair of the squashed S^3 LLSM [57],

$$\begin{aligned}U(t, x; z) &= \frac{\alpha}{\sin z} [n^1 T^1 + n^2 T^2 + \cos z n^3 T^3], \\ V(t, x; z) &= \frac{\beta}{\sin z} [(n^2 \partial_x n^3 - n^3 \partial_x n^2) T^1 + (n^3 \partial_x n^1 - n^1 \partial_x n^3) T^2 \\ &\quad + \cos z (n^1 \partial_x n^2 - n^2 \partial_x n^1) T^3] \\ &\quad - \frac{\alpha\beta}{\sin^2 z} [\cos z n^1 T^1 + \cos z n^2 T^2 + n^3 T^3],\end{aligned}\tag{B.7}$$

where we have introduced a spectral parameter $z \in \mathbb{C}$ and new parameters,

$$\alpha \equiv \frac{4\pi L}{\sqrt{\lambda}} \sqrt{C}, \quad \beta \equiv \frac{\sqrt{\lambda}}{2\pi L} \sqrt{C}.\tag{B.8}$$

It is an easy task to check that the equations of motion (B.4) are reproduced from the commutation relation

$$[\partial_t - V(t, x; z), \partial_x - U(t, x; z)] = 0.\tag{B.9}$$

The monodromy matrix can be introduced as

$$M(z) \equiv P \exp \left[\int_{-\infty}^{\infty} dx U(t, x; z) \right],\tag{B.10}$$

where P denotes the parth-ordering. This is a conserved quantity again. As we will see later, the expansions around $z = \pm\infty$ lead to a quantum affine algebra $U_q(\widehat{su(2)})$.

B.2 The standard q -deformation of $su(2)$

Let us consider a q -deformation of $su(2)$ realized in the squashed S^3 LLSM.

The $SU(2)$ symmetry of the isotropic S^3 LLSM is broken to $U(1)$ due to the deformation. The remaining $U(1)$ generator is given by

$$Q^3 = -\frac{L}{2} \int_{-\infty}^{\infty} dx n^3(x).\tag{B.11}$$

¹Here we have introduced the totally anti-symmetric tensor ε_{abc} with $\varepsilon_{123} = +1$.

Note here that the broken components of $SU(2)$ are still realized in a non-local way even when $C \neq 0$, as in the time-like warped $SL(2)$ LLSM.

To find out the corresponding non-local charges, it is helpful to introduce n^\pm defined as

$$n^\pm \equiv \frac{n^1 \pm in^2}{\sqrt{2}}. \quad (\text{B.12})$$

We take the rapidly damping condition so that n^\pm vanish at the spatial infinities. The non-local charges are conserved under this boundary condition (See Appendix C).

The non-local conserved charges are given by

$$Q^\pm = -\frac{L}{2} \int_{-\infty}^{\infty} dx e^{\alpha\chi(x)} n^\pm(x), \quad (\text{B.13})$$

where $\chi(x)$ is a non-local field defined as

$$\chi(x) \equiv \frac{1}{2} \int_{-\infty}^{\infty} dy \epsilon(x-y) n^3(y). \quad (\text{B.14})$$

Here the following relations are useful to show the conservation laws of Q^\pm ,

$$\partial_x \chi = n^3, \quad \partial_t \chi = i \frac{\lambda}{8\pi^2 L^2} (n^+ \partial_x n^- - n^- \partial_x n^+). \quad (\text{B.15})$$

The next is to compute the Poisson brackets of Q^3 and Q^\pm . The brackets for n^a ($a = 3, \pm$) are

$$\{n^a(x), n^b(y)\}_P = -i \frac{2}{L} \epsilon^{ab}{}_c n^c(x) \delta(x-y). \quad (\text{B.16})$$

Non-ultra local terms are not contained again.

With the brackets (B.16), the brackets of Q^0 and Q^\pm can be evaluated as

$$\begin{aligned} \{Q^\pm, Q^3\}_P &= \pm i Q^\pm, \\ \{Q^+, Q^-\}_P &= -i \frac{L}{2\alpha} \sinh\left(\frac{2\alpha}{L} Q^3\right). \end{aligned} \quad (\text{B.17})$$

This is a classical analogue of the standard q -deformation of $su(2)$ [32, 33].

In addition, there exists another set of non-local conserved charges,

$$\tilde{Q}^\pm = -\frac{L}{2} \int_{-\infty}^{\infty} dx e^{-\alpha\chi(x)} n^\pm(x). \quad (\text{B.18})$$

These are obtained by changing the sign of α in Q^\pm . Hence, by construction, Q^3 and \tilde{Q}^\pm also generate another q -deformed $su(2)$ algebra,

$$\begin{aligned} \{\tilde{Q}^\pm, Q^3\}_P &= \pm i \tilde{Q}^\pm, \\ \{\tilde{Q}^+, \tilde{Q}^-\}_P &= -i \frac{L}{2\alpha} \sinh\left(\frac{2\alpha}{L} Q^3\right). \end{aligned} \quad (\text{B.19})$$

It is possible to show that a quantum affine algebra is generated by Q^3 , Q^\pm and \tilde{Q}^\pm in the sense of Drinfeld's first realization [31]. The derivation is essentially the same as in the time-like warped LLSM, hence we will not give the derivation.

B.3 The classical r -matrix

Finally, let us discuss the classical r -matrix from (3.34). The resulting classical r -matrix is of trigonometric type [57],

$$r(z-w) = \frac{2}{L} \frac{\alpha}{\sin(z-w)} \left[T^1 \otimes T^1 + T^2 \otimes T^2 + \cos(z-w) T^3 \otimes T^3 \right], \quad (\text{B.20})$$

and satisfies the classical Yang-Baxter equation (3.36). This ensures the classical integrability of the squashed S^3 LLSM.

C Boundary conditions and conservation laws

We argue here the relation between boundary conditions and the conservation laws of non-local charges for the time-like warped $SL(2)$ LLSM, the squashed S^3 LLSM and the null-like warped $SL(2)$ LLSM.

C.1 Time-like warped $SL(2)$ LLSM

Let us first consider the time-like warped $SL(2)$ LLSM.

We impose the following boundary condition:

$$n^{\hat{\pm}}(x) \rightarrow 0 \quad \text{as} \quad x \rightarrow \pm\infty. \quad (\text{C.1})$$

In terms of the coordinates, the boundary condition (C.1) is expressed as

$$\rho(x) \rightarrow 0, \quad \psi(x) \rightarrow \text{const.} \quad \text{as} \quad x \rightarrow \pm\infty. \quad (\text{C.2})$$

By using the relation

$$(n^0)^2 - 2n^{\hat{+}}n^{\hat{-}} = 1, \quad (\text{C.3})$$

the condition (C.1) implies that $(n^0)^2 \rightarrow 1$ as $x \rightarrow \pm\infty$. Because we are working with the parametrization $n^0 = -\cosh \rho$, the boundary condition for n^0 is fixed as follows:

$$n^0(x) \rightarrow -1 \quad \text{as} \quad x \rightarrow \pm\infty. \quad (\text{C.4})$$

It should be noted here that the local charge Q^0 is not finite and diverges under the condition (C.4). However, this divergence may have a physical interpretation. The value of Q^0 is closely related to the length of the spatial direction x of the string world-sheet due to the spin chain interpretation. We are now considering the infinite spatial direction of the string world-sheet, hence the divergence of Q^0 may be interpreted as a rather natural result.

Let us next check the conservation laws of $Q^{\hat{\pm}}$. One can show them as follows:

$$\partial_t Q^{\hat{+}} = -\frac{L}{2} \int_{-\infty}^{\infty} dx \partial_t [e^{\alpha x(x)} n^{\hat{+}}(x)]$$

$$\begin{aligned}
&= -i \frac{\lambda}{16\pi^2 L} \int_{-\infty}^{\infty} dx \partial_x [e^{\alpha\chi} (n^{\hat{+}} \partial_x n^0 - n^0 \partial_x n^{\hat{+}})] - i \frac{\sqrt{\lambda}}{4\pi} \sqrt{C} \int_{-\infty}^{\infty} dx \partial_x [e^{\alpha\chi} n^{\hat{+}}] \\
&= 0, \\
\partial_t Q^{\hat{-}} &= -\frac{L}{2} \int_{-\infty}^{\infty} dx \partial_t [e^{\alpha\chi(x)} n^{\hat{-}}(x)] \\
&= -i \frac{\lambda}{16\pi^2 L} \int_{-\infty}^{\infty} dx \partial_x [e^{\alpha\chi} (n^0 \partial_x n^{\hat{-}} - n^{\hat{-}} \partial_x n^0)] + i \frac{\sqrt{\lambda}}{4\pi} \sqrt{C} \int_{-\infty}^{\infty} dx \partial_x [e^{\alpha\chi} n^{\hat{-}}] \\
&= 0.
\end{aligned}$$

Under the condition (C.1), all of the surface terms have vanished.

Similarly, for another set of non-local charges $\tilde{Q}^{\hat{\pm}}$, the conservation laws are shown as

$$\begin{aligned}
\partial_t \tilde{Q}^{\hat{+}} &= -\frac{L}{2} \int_{-\infty}^{\infty} dx \partial_t [e^{-\alpha\chi(x)} n^{\hat{+}}(x)] \\
&= -i \frac{\lambda}{16\pi^2 L} \int_{-\infty}^{\infty} dx \partial_x [e^{-\alpha\chi} (n^{\hat{+}} \partial_x n^0 - n^0 \partial_x n^{\hat{+}})] + i \frac{\sqrt{\lambda}}{4\pi} \sqrt{C} \int_{-\infty}^{\infty} dx \partial_x [e^{-\alpha\chi} n^{\hat{+}}] \\
&= 0, \\
\partial_t \tilde{Q}^{\hat{-}} &= -\frac{L}{2} \int_{-\infty}^{\infty} dx \partial_t [e^{-\alpha\chi(x)} n^{\hat{-}}(x)] \\
&= -i \frac{\lambda}{16\pi^2 L} \int_{-\infty}^{\infty} dx \partial_x [e^{-\alpha\chi} (n^0 \partial_x n^{\hat{-}} - n^{\hat{-}} \partial_x n^0)] - i \frac{\sqrt{\lambda}}{4\pi} \sqrt{C} \int_{-\infty}^{\infty} dx \partial_x [e^{-\alpha\chi} n^{\hat{-}}] \\
&= 0.
\end{aligned}$$

Thus $\tilde{Q}^{\hat{\pm}}$ are also conserved under the condition (C.1).

Note that all of the non-local charges are finite under the condition (C.1), while the local charge Q^0 diverges.

C.2 Squashed S^3 LLSM

Next, we consider the squashed S^3 LLSM. The argument here is essentially the same as in the case of the time-like warped $SL(2)$ LLSM.

First of all, let us impose the following boundary condition:

$$n^{\pm}(x) \rightarrow 0 \quad \text{as} \quad x \rightarrow \pm\infty. \quad (\text{C.5})$$

In terms of the coordinates, the condition (C.5) is expressed as

$$\theta(x) \rightarrow 0, \quad \phi(x) \rightarrow \text{const.} \quad \text{as} \quad x \rightarrow \pm\infty. \quad (\text{C.6})$$

By using the relation

$$(n^3)^2 + 2n^+ n^- = 1, \quad (\text{C.7})$$

the condition (C.5) indicates that $(n^3)^2 \rightarrow 1$ as $x \rightarrow \pm\infty$. Due to the parametrization $n^3 = \cos \theta$, the boundary condition for n^3 is fixed as follows:

$$n^3(x) \rightarrow 1 \quad \text{as} \quad x \rightarrow \pm\infty. \quad (\text{C.8})$$

As a result, the local charge Q^3 is not finite but diverges again. The divergence may also be interpreted physically, as mentioned before.

The conservation laws of Q^\pm are shown as

$$\begin{aligned}
\partial_t Q^+ &= -\frac{L}{2} \int_{-\infty}^{\infty} dx \partial_t [e^{\alpha\chi(x)} n^+(x)] \\
&= -i \frac{\lambda}{16\pi^2 L} \int_{-\infty}^{\infty} dx \partial_x [e^{\alpha\chi} (n^3 \partial_x n^+ - n^+ \partial_x n^3)] + i \frac{\sqrt{\lambda}}{4\pi} \sqrt{C} \int_{-\infty}^{\infty} dx \partial_x [e^{\alpha\chi} n^+] \\
&= 0, \\
\partial_t Q^- &= -\frac{L}{2} \int_{-\infty}^{\infty} dx \partial_t [e^{\alpha\chi(x)} n^-(x)] \\
&= -i \frac{\lambda}{16\pi^2 L} \int_{-\infty}^{\infty} dx \partial_x [e^{\alpha\chi} (n^- \partial_x n^3 - n^3 \partial_x n^-)] - i \frac{\sqrt{\lambda}}{4\pi} \sqrt{C} \int_{-\infty}^{\infty} dx \partial_x [e^{\alpha\chi} n^-] \\
&= 0.
\end{aligned}$$

All of the surface terms have vanished under the condition (C.5).

Also for \tilde{Q}^\pm , the conservation laws are shown as follows:

$$\begin{aligned}
\partial_t \tilde{Q}^+ &= -\frac{L}{2} \int_{-\infty}^{\infty} dx \partial_t [e^{-\alpha\chi(x)} n^{\hat{+}}(x)] \\
&= -i \frac{\lambda}{16\pi^2 L} \int_{-\infty}^{\infty} dx \partial_x [e^{\alpha\chi} (n^3 \partial_x n^+ - n^+ \partial_x n^3)] - i \frac{\sqrt{\lambda}}{4\pi} \sqrt{C} \int_{-\infty}^{\infty} dx \partial_x [e^{-\alpha\chi} n^+] \\
&= 0, \\
\partial_t \tilde{Q}^- &= -\frac{L}{2} \int_{-\infty}^{\infty} dx \partial_t [e^{-\alpha\chi(x)} n^{\hat{-}}(x)] \\
&= -i \frac{\lambda}{16\pi^2 L} \int_{-\infty}^{\infty} dx \partial_x [e^{\alpha\chi} (n^- \partial_x n^3 - n^3 \partial_x n^-)] + i \frac{\sqrt{\lambda}}{4\pi} \sqrt{C} \int_{-\infty}^{\infty} dx \partial_x [e^{-\alpha\chi} n^-] \\
&= 0.
\end{aligned}$$

Thus \tilde{Q}^\pm are conserved.

Note that all of the non-local charges are finite, while the local charge diverges.

C.3 Null-like warped $SL(2)$ LLSM

Finally, we consider the null-like warped $SL(2)$ LLSM.

For this case, let us introduce the following boundary condition:

$$n^-(x), \quad n^2(x) \rightarrow 0 \quad \text{as} \quad x \rightarrow \pm\infty. \quad (\text{C.9})$$

In terms of the coordinates, this condition (C.9) is realized as the condition,

$$\rho(x) \rightarrow \infty, \quad \psi(x) \rightarrow \frac{3}{2}\pi \quad \text{as} \quad x \rightarrow \pm\infty, \quad (\text{C.10})$$

where ρ is supposed to diverge logarithmically. From the relation

$$2n^+ n^- - (n^2)^2 = 1, \quad (\text{C.11})$$

the condition (C.9) indicates that n^+ should diverge as $x \rightarrow \pm\infty$.

$$n^+(x) \rightarrow -\infty \quad \text{as} \quad x \rightarrow \pm\infty. \quad (\text{C.12})$$

Then the non-local charges Q^+ and \tilde{Q}^+ are not finite and diverge. Hence a different component of the non-local charges diverges in comparison to the previous two cases. The difference comes from a pp-wave like limit and the divergence may be basically interpreted as an infinite length of the spatial direction of the string world-sheet.

One can check that Q^+ and Q^2 are conserved as follows:

$$\begin{aligned} \partial_t Q^+ &= -\frac{L}{2} \int_{-\infty}^{\infty} dx \partial_t [e^{\sqrt{C}\alpha\chi(x)} n^+(x)] \\ &= \frac{\lambda}{16\pi^2 L} \int_{-\infty}^{\infty} dx \partial_x [e^{\sqrt{C}\alpha\chi} (n^2 \partial_x n^+ - n^+ \partial_x n^2)] + \frac{\sqrt{\lambda}}{4\pi} \sqrt{C} \int_{-\infty}^{\infty} dx \partial_x [e^{\sqrt{C}\alpha\chi} n^2] \\ &= 0, \\ \partial_t Q^2 &= -\frac{L}{2} \int_{-\infty}^{\infty} dx \partial_t [e^{\sqrt{C}\alpha\chi(x)} n^2(x)] \\ &= \frac{\lambda}{16\pi^2 L} \int_{-\infty}^{\infty} dx \partial_x [e^{\sqrt{C}\alpha\chi} (n^- \partial_x n^+ - n^+ \partial_x n^-)] + \frac{\sqrt{\lambda}}{4\pi} \sqrt{C} \int_{-\infty}^{\infty} dx \partial_x [e^{\sqrt{C}\alpha\chi} n^-] \\ &= 0. \end{aligned}$$

Under the condition (C.9), all of the surface terms have vanished. Note that the differential terms are ensured to vanish due to the logarithmic behavior of ρ around the spatial infinities.

The conservation laws of \tilde{Q}^+ and \tilde{Q}^2 are also shown as follows:

$$\begin{aligned} \partial_t \tilde{Q}^+ &= -\frac{L}{2} \int_{-\infty}^{\infty} dx \partial_t [e^{-\sqrt{C}\alpha\chi(x)} n^+(x)] \\ &= \frac{\lambda}{16\pi^2 L} \int_{-\infty}^{\infty} dx \partial_x [e^{-\sqrt{C}\alpha\chi} (n^2 \partial_x n^+ - n^+ \partial_x n^2)] - \frac{\sqrt{\lambda}}{4\pi} \sqrt{C} \int_{-\infty}^{\infty} dx \partial_x [e^{-\sqrt{C}\alpha\chi} n^2] \\ &= 0, \\ \partial_t \tilde{Q}^2 &= -\frac{L}{2} \int_{-\infty}^{\infty} dx \partial_t [e^{-\sqrt{C}\alpha\chi(x)} n^2(x)] \\ &= \frac{\lambda}{16\pi^2 L} \int_{-\infty}^{\infty} dx \partial_x [e^{-\sqrt{C}\alpha\chi} (n^- \partial_x n^+ - n^+ \partial_x n^-)] - \frac{\sqrt{\lambda}}{4\pi} \sqrt{C} \int_{-\infty}^{\infty} dx \partial_x [e^{-\sqrt{C}\alpha\chi} n^-] \\ &= 0. \end{aligned}$$

As a result, \tilde{Q}^+ and \tilde{Q}^2 are conserved under the condition (C.9).

Note that the non-local charges Q^2 and \tilde{Q}^2 as well as the local charge Q^- are finite. However, the non-local charges Q^+ and \tilde{Q}^+ diverge.

D Null-like warped LLSM from time-like LLSM

We derive here the null-like warped $SL(2)$ LLSM from the time-like warped $SL(2)$ LLSM by taking an appropriate scaling-limit (a pp-wave like limit). The classical equations of motion are first reproduced and then the Lax pair is also derived.

D.1 Equations of motion

The equations of motion of the null-like warped $SL(2)$ LLSM are reproduced from those of the time-like warped $SL(2)$ LLSM by taking a scaling limit.

Recall the equations of motion for the time-like warped $SL(2)$ LLSM,

$$\begin{aligned}\partial_t n^0 &= \frac{\lambda}{8\pi^2 L^2} (n^1 \partial_x^2 n^2 - n^2 \partial_x^2 n^1) , \\ \partial_t n^1 &= \frac{\lambda}{8\pi^2 L^2} (n^0 \partial_x^2 n^2 - n^2 \partial_x^2 n^0) - 2\tilde{C} n^0 n^2 , \\ \partial_t n^2 &= \frac{\lambda}{8\pi^2 L^2} (n^1 \partial_x^2 n^0 - n^0 \partial_x^2 n^1) + 2\tilde{C} n^0 n^1 .\end{aligned}\tag{D.1}$$

Here we have used \tilde{C} as a deformation parameter. It is convenient to introduce n^\pm as

$$n^\pm = \frac{n^0 \pm n^1}{\sqrt{2}} .\tag{D.2}$$

Then let us rewrite the equations of motion in terms of n^\pm with the following rescaling:

$$n^- \rightarrow \sqrt{\frac{2C}{\tilde{C}}} n^- , \quad n^+ \rightarrow \sqrt{\frac{\tilde{C}}{2C}} n^+ .\tag{D.3}$$

Finally, by taking the $\tilde{C} \rightarrow 0$ limit with C fixed, the equations of motion are evaluated as

$$\begin{aligned}\partial_t n^- &= -\frac{\lambda}{8\pi^2 L^2} (n^- \partial_x^2 n^2 - n^2 \partial_x^2 n^-) , \\ \partial_t n^2 &= -\frac{\lambda}{8\pi^2 L^2} (n^- \partial_x^2 n^+ - n^+ \partial_x^2 n^-) - 2C (n^-)^2 , \\ \partial_t n^+ &= -\frac{\lambda}{8\pi^2 L^2} (n^2 \partial_x^2 n^+ - n^+ \partial_x^2 n^2) - 2C n^- n^2 .\end{aligned}\tag{D.4}$$

These are equivalent to the equations of motion for the null-like warped LLSM in (4.3).

D.2 A Lax pair of null-like warped $SL(2)$ LLSM

Let us derive the Lax pair of the null-like warped $SL(2)$ LLSM.

We start from the Lax pair of the time-like warped $SL(2)$ LLSM.

$$\begin{aligned}U(t, x; z) &= \frac{i\tilde{\alpha}}{\sinh z} [-\cosh z n^0 T^0 + n^1 T^1 + n^2 T^2] , \\ V(t, x; z) &= \frac{i\tilde{\beta}}{\sinh z} [-\cosh z (n^1 \partial_x n^2 - n^2 \partial_x n^1) T^0 + (n^0 \partial_x n^2 - n^2 \partial_x n^0) T^1 \\ &\quad + (n^1 \partial_x n^0 - n^0 \partial_x n^1) T^2] \\ &\quad + \frac{\tilde{\alpha}\tilde{\beta}}{\sinh^2 z} [-n^0 T^0 + \cosh z n^1 T^1 + \cosh z n^2 T^2] .\end{aligned}\tag{D.5}$$

Here $\tilde{\alpha}$ and $\tilde{\beta}$ are defined as

$$\tilde{\alpha} \equiv \frac{4\pi L}{\sqrt{\lambda}} \sqrt{\tilde{C}} , \quad \tilde{\beta} \equiv \frac{\sqrt{\lambda}}{2\pi L} \sqrt{\tilde{C}} ,\tag{D.6}$$

and \tilde{C} is a deformation parameter.

Let us rewrite the Lax pair (D.5) in terms of T^\pm and then rescale z , n^\pm and T^\pm like

$$\begin{aligned} z &\rightarrow i\sqrt{\tilde{C}} z, \quad n^- \rightarrow \sqrt{\frac{2C}{\tilde{C}}} n^-, \quad n^+ \rightarrow \sqrt{\frac{\tilde{C}}{2C}} n^+, \\ T^- &\rightarrow \sqrt{\frac{2C}{\tilde{C}}} T^-, \quad T^+ \rightarrow \sqrt{\frac{\tilde{C}}{2C}} T^+. \end{aligned} \quad (\text{D.7})$$

Finally, by taking the $\tilde{C} \rightarrow 0$ limit, the following Lax pair is obtained,

$$\begin{aligned} U(t, x; z) &= \frac{\alpha}{z} \left(- \left[n^+ - \frac{Cz^2}{2} n^- \right] T^- + n^2 T^2 - n^- T^+ \right), \\ V(t, x; z) &= \frac{\beta}{z} \left(- \left[(n^2 \partial_x n^+ - n^+ \partial_x n^2) - \frac{Cz^2}{2} (n^- \partial_x n^2 - n^2 \partial_x n^-) \right] T^- \right. \\ &\quad \left. + (n^- \partial_x n^+ - n^+ \partial_x n^-) T^2 - (n^- \partial_x n^2 - n^2 \partial_x n^-) T^+ \right) \\ &\quad + \frac{\alpha\beta}{z^2} \left(- \left[n^+ + \frac{Cz^2}{2} n^- \right] T^- + n^2 T^2 - n^- T^+ \right). \end{aligned} \quad (\text{D.8})$$

Here α and β have newly been introduced as

$$\alpha = \frac{4\pi L}{\sqrt{\lambda}}, \quad \beta = -\frac{\sqrt{\lambda}}{2\pi L}. \quad (\text{D.9})$$

From the Lax pair (D.8), one can reproduce the equations of motion for the null-like warped $SL(2)$ LLSM obtained in [54].

E Undoing Jordanian twists

We present here non-local gauge transformations for the Lax pair of the null-like warped $SL(2)$ LLSM. The gauge transformations may be interpreted as undoing Jordanian twists.

Let us start from the following Lax pair,

$$\begin{aligned} U(t, x; z) &= \frac{\alpha}{z} \left(- \left[n^+ - \frac{Cz^2}{2} n^- \right] T^- + n^2 T^2 - n^- T^+ \right), \\ V(t, x; z) &= \frac{\beta}{z} \left(- \left[(n^2 \partial_x n^+ - n^+ \partial_x n^2) - \frac{Cz^2}{2} (n^- \partial_x n^2 - n^2 \partial_x n^-) \right] T^- \right. \\ &\quad \left. + (n^- \partial_x n^+ - n^+ \partial_x n^-) T^2 - (n^- \partial_x n^2 - n^2 \partial_x n^-) T^+ \right) \\ &\quad + \frac{\alpha\beta}{z^2} \left(- \left[n^+ + \frac{Cz^2}{2} n^- \right] T^- + n^2 T^2 - n^- T^+ \right). \end{aligned} \quad (\text{E.1})$$

Note that this Lax pair has two poles at $z = 0$ and $z = \infty$.

The isomorphisms of $sl(2)$ algebra,

$$T^- \rightarrow T^-, \quad T^2 \rightarrow T^2 \mp \sqrt{C}zT^-, \quad T^+ \rightarrow T^+ \mp \sqrt{C}zT^2 + \frac{Cz^2}{2}T^-, \quad (\text{E.2})$$

transforms the Lax pair (E.1) into

$$\begin{aligned} U^{(\pm)}(t, x; z) &= \frac{\alpha}{z} \left[- \left(n^+ \pm \sqrt{C}zn^2 \right) T^- + \left(n^2 \pm \sqrt{C}zn^- \right) T^2 - n^- T^+ \right], \\ V^{(\pm)}(t, x; z) &= \frac{\beta}{z} \left(- \left[\left(n^2 \partial_x n^+ - n^+ \partial_x n^2 \right) \pm \sqrt{C}z \left(n^- \partial_x n^+ - n^+ \partial_x n^- \right) \right] T^- \right. \\ &\quad + \left[\left(n^- \partial_x n^+ - n^+ \partial_x n^- \right) \pm \sqrt{C}z \left(n^- \partial_x n^2 - n^2 \partial_x n^- \right) \right] T^2 \\ &\quad \left. - \left(n^- \partial_x n^2 - n^2 \partial_x n^- \right) T^+ \right) \\ &\quad + \frac{\alpha\beta}{z^2} \left(- \left[n^+ \pm \sqrt{C}zn^2 + Cz^2n^- \right] T^- + \left(n^2 \pm \sqrt{C}zn^- \right) T^2 - n^- T^+ \right). \end{aligned} \quad (\text{E.3})$$

The resulting Lax pairs $(U^{(\pm)}(z), V^{(\pm)}(z))$ do not diverge at $z = \infty$ any more. Then the asymptotic expressions are given by

$$\begin{aligned} U^{(\pm)}(t, x; z = \infty) &= \pm \sqrt{C}\alpha \left(-n^2 T^- + n^- T^2 \right), \\ V^{(\pm)}(t, x; z = \infty) &= \pm \sqrt{C}\beta \left[- \left(n^- \partial_x n^+ - n^+ \partial_x n^- \right) T^- \right. \\ &\quad \left. + \left(n^- \partial_x n^2 - n^2 \partial_x n^- \right) T^2 \right] - C\alpha\beta n^- T^-. \end{aligned} \quad (\text{E.4})$$

These are useful to construct non-local gauge transformations.

With the help of (E.4), non-local functions can be defined as

$$\mathcal{F}^{(\pm)}(t, x) \equiv \text{P exp} \left[\int_{-\infty}^x dy U^{(\pm)}(t, y; z = \infty) \right] K^{(\pm)}. \quad (\text{E.5})$$

These functions are obtained as the solutions of the following differential equations:

$$\partial_t \mathcal{F}^{(\pm)} = V^{(\pm)}(z = \infty) \mathcal{F}^{(\pm)}, \quad \partial_x \mathcal{F}^{(\pm)} = U^{(\pm)}(z = \infty) \mathcal{F}^{(\pm)}. \quad (\text{E.6})$$

When solving the differential equations, $K^{(\pm)}$ are introduced as integral constant matrices. It is convenient to take $K^{(\pm)}$ as follows:

$$K^{(\pm)} = \exp \left[\pm \frac{2\sqrt{C}\alpha}{L} Q^- T^2 \right]. \quad (\text{E.7})$$

These choices are basically fixed in [25] by borrowing the knowledge of quantum Jordanian twists. Then the explicit expressions of $\mathcal{F}^{(\pm)}$ are given by, respectively,

$$\begin{aligned} \mathcal{F}^{(+)} &= \exp \left[\sqrt{C}\alpha \chi(x) T^2 \right] \exp \left[-\sqrt{C}\alpha \left(\xi(x) - \frac{1}{L} Q^2 \right) T^- \right] \exp \left[\frac{\sqrt{C}\alpha}{L} Q^- T^2 \right], \\ \mathcal{F}^{(-)} &= \exp \left[-\sqrt{C}\alpha \chi(x) T^2 \right] \exp \left[\sqrt{C}\alpha \left(\tilde{\xi}(x) - \frac{1}{L} \tilde{Q}^2 \right) T^- \right] \exp \left[-\frac{\sqrt{C}\alpha}{L} Q^- T^2 \right]. \end{aligned} \quad (\text{E.8})$$

where $\xi(x)$ and $\tilde{\xi}(x)$ are non-local fields defined as

$$\begin{aligned}\xi(x) &\equiv \frac{1}{2} \int_{-\infty}^{\infty} dy \, \epsilon(x-y) e^{\sqrt{C}\alpha\chi(x)} n^2(y), \\ \tilde{\xi}(x) &\equiv \frac{1}{2} \int_{-\infty}^{\infty} dy \, \epsilon(x-y) e^{-\sqrt{C}\alpha\chi(x)} n^2(y).\end{aligned}\tag{E.9}$$

The next is to see the transformation laws of the Lax pairs $(U^{(\pm)}(z), V^{(\pm)}(z))$ under gauge transformations generated by $\mathcal{F}^{(\pm)}$,

$$\begin{aligned}\mathcal{U}^{(\pm)}(z) &= \left(\mathcal{F}^{(\pm)}\right)^{-1} U(z) \mathcal{F}^{(\pm)} - \left(\mathcal{F}^{(\pm)}\right)^{-1} \partial_x \mathcal{F}^{(\pm)}, \\ \mathcal{V}^{(\pm)}(z) &= \left(\mathcal{F}^{(\pm)}\right)^{-1} V(z) \mathcal{F}^{(\pm)} - \left(\mathcal{F}^{(\pm)}\right)^{-1} \partial_t \mathcal{F}^{(\pm)}.\end{aligned}\tag{E.10}$$

The resulting Lax pairs $(\mathcal{U}^{(\pm)}(z), \mathcal{V}^{(\pm)}(z))$ are explicitly given by

$$\begin{aligned}\mathcal{U}^{(+)}(z) &= \frac{\alpha}{z} \left[-\mathcal{N}^- T^+ + \mathcal{N}^2 T^2 - \mathcal{N}^+ T^- \right], \\ \mathcal{V}^{(+)}(z) &= \frac{\beta}{z} \left[-(\mathcal{N}^- \partial_x \mathcal{N}^2 - \mathcal{N}^2 \partial_x \mathcal{N}^-) T^+ + (\mathcal{N}^- \partial_x \mathcal{N}^+ - \mathcal{N}^+ \partial_x \mathcal{N}^-) T^2 \right. \\ &\quad \left. - (\mathcal{N}^2 \partial_x \mathcal{N}^+ - \mathcal{N}^+ \partial_x \mathcal{N}^2) T^- \right] + \frac{\alpha\beta}{z^2} \left[-\mathcal{N}^- T^+ + \mathcal{N}^2 T^2 - \mathcal{N}^+ T^- \right], \\ \mathcal{U}^{(-)}(z) &= \frac{\alpha}{z} \left[-\tilde{\mathcal{N}}^- T^+ + \tilde{\mathcal{N}}^2 T^2 - \tilde{\mathcal{N}}^+ T^- \right], \\ \mathcal{V}^{(-)}(z) &= \frac{\beta}{z} \left[-(\tilde{\mathcal{N}}^- \partial_x \tilde{\mathcal{N}}^2 - \tilde{\mathcal{N}}^2 \partial_x \tilde{\mathcal{N}}^-) T^+ + (\tilde{\mathcal{N}}^- \partial_x \tilde{\mathcal{N}}^+ - \tilde{\mathcal{N}}^+ \partial_x \tilde{\mathcal{N}}^-) T^2 \right. \\ &\quad \left. - (\tilde{\mathcal{N}}^2 \partial_x \tilde{\mathcal{N}}^+ - \tilde{\mathcal{N}}^+ \partial_x \tilde{\mathcal{N}}^2) T^- \right] + \frac{\alpha\beta}{z^2} \left\{ -\tilde{\mathcal{N}}^- T^+ + \tilde{\mathcal{N}}^2 T^2 - \tilde{\mathcal{N}}^+ T^- \right\},\end{aligned}\tag{E.11}$$

where the components of non-local vectors \mathcal{N}^a and $\tilde{\mathcal{N}}^a$ are given by

$$\begin{aligned}\mathcal{N}^- &= -\frac{1}{\sqrt{C}\alpha} e^{-\frac{\sqrt{C}\alpha}{L} Q^-} \partial_x e^{-\sqrt{C}\alpha\chi(x)}, \quad \mathcal{N}^2 = \partial_x \left[\left(\xi - \frac{1}{L} Q^2 \right) e^{-\sqrt{C}\alpha\chi(x)} \right], \\ \mathcal{N}^+ &= e^{\frac{\sqrt{C}\alpha}{L} Q^-} \left(e^{\sqrt{C}\alpha\chi(x)} n^+(x) - \frac{\sqrt{C}\alpha}{2} \partial_x \left[\left(\xi - \frac{1}{L} Q^2 \right)^2 e^{-\sqrt{C}\alpha\chi(x)} \right] \right), \\ \tilde{\mathcal{N}}^- &= \frac{1}{\sqrt{C}\alpha} e^{\frac{\sqrt{C}\alpha}{L} Q^-} \partial_x e^{\sqrt{C}\alpha\chi(x)}, \quad \tilde{\mathcal{N}}^2 = \partial_x \left[\left(\tilde{\xi} - \frac{1}{L} \tilde{Q}^2 \right) e^{\sqrt{C}\alpha\chi(x)} \right], \\ \tilde{\mathcal{N}}^+ &= e^{-\frac{\sqrt{C}\alpha}{L} Q^-} \left(e^{-\sqrt{C}\alpha\chi(x)} n^+(x) + \frac{\sqrt{C}\alpha}{2} \partial_x \left[\left(\tilde{\xi} - \frac{1}{L} \tilde{Q}^2 \right)^2 e^{\sqrt{C}\alpha\chi(x)} \right] \right).\end{aligned}\tag{E.12}$$

They satisfy the following relations,

$$2\mathcal{N}^- \mathcal{N}^+ - (\mathcal{N}^2)^2 = 1, \quad 2\tilde{\mathcal{N}}^- \tilde{\mathcal{N}}^+ - (\tilde{\mathcal{N}}^2)^2 = 1.\tag{E.13}$$

The monodromy matrices constructed from the Lax pairs $(\mathcal{U}^{(\pm)}(z), \mathcal{V}^{(\pm)}(z))$ lead to a couple of Yangians, as shown in the body of this manuscript. This result indicates that the non-local gauge transformations are nothing but undoing Jordanian twists.

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